Stochastic models in neuroscience:

Poisson point process simulation approach

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LASCON VIII
Membrane potential and spikes

Membrane potential = difference in voltage between the inner and outer cellular membrane
Membrane potential and spikes

\[
\text{Membrane potential} = V(t) \in \mathbb{R}
\]
Membrane potential and spikes

Membrane potential \( V(t) \in \mathbb{R} \)

Temporal evolution

1. A neuron accumulate potential duo to a stimuli current \( I(t) \)
   (interaction with other neurons ou external stimuli)

2. The occurrence of a spike in a neuron depends on its membrane potential (the higher the potential the higher the spiking probability)

3. When a neuron spikes, it loses all the accumulated potential
Single integrate-and-fire model

**Deterministic version:** the membrane potential
- integrate a current up to a threshold
- “spike”
- reset the membrane potential to a resting value
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**Stochastic version:** the membrane potential
- integrate a current up to a **stochastic** threshold
- “spike”
- reset the membrane potential to a resting value
Single integrate-and-fire model

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**Stochastic version:** the membrane potential
- integrate a current up to a stochastic threshold
  - the higher the membrane potential the higher the spiking probability
- “spike”
- reset the membrane potential to a resting value
Formally

Let $C(t)$ be a stimuli current

**Deterministic version:**

Fix a threshold $\nu$ and denote by $L = \sup\{s < t : V(s) = V_{rest}\}$, then

$$V(t) = \int_{L}^{t} C(s)ds \quad \text{up to a threshold } \nu.$$

If $V(t^-) = \nu$, then set $V(t) = V_{rest}$. 
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$$V(t) = \int_{L}^{t} C(s) \, ds \quad \text{up to a threshold } \nu,$$

Then

$$P(V(s) = V_{\text{rest}} \text{ for some } s \in [t, t + dt] \mid V(t))$$

depends on $V(t) \, dt$
Formally

Let $C(t)$ be a stimuli current

**Stochastic version:** Denote by $L = \sup\{ s < t : V(s) = V_{\text{rest}} \}$ and consider

$$V(t) = \int_{L}^{t} C(s) ds$$

up to a threshold $\nu$.

Then

$$P( V(s) = V_{\text{rest}} \text{ for some } s \in [t, t + dt] | V(t) ) \approx \varphi(V(t)) dt$$

with $\varphi : \mathbb{R} \rightarrow [0, 1]$ a given spiking rate function.
How can we do it?

Poisson Point Process
One Dimensional Poisson Point Process

Denote by $T_k, k \geq 1$ the spiking times of the neuron
For any interval $A \cup \mathbb{R}$ consider $N(A)$ the number of spikes which occurred in $A$. 
One Dimensional Poisson Point Process

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A Poisson point process $N$ with constant rate $\lambda$ satisfies

(i) For each interval $A \cup \mathbb{R}$, $N(A)$ has Poisson distribution

$$P(N(A) = k) = \frac{e^{-\lambda|A|}(\lambda|A|)^k}{k!} \quad k = 0, 1, \ldots$$
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(iv) Given the spiking times \( (T_k)_{k} \) of a Poisson process, the interspike interval \( T_k - T_{k-1} \) has Exponential distribution of parameter \( \lambda \)
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(iv) Given the spiking times $(T_k)_k$ of a Poisson process, the interspike interval $T_k - T_{k-1}$ has Exponential distribution of parameter $\lambda$.

(v) $E[N(A)] = \lambda|A|$ (in average $\lambda$ spikes per $(r, r+1]$ interval)
Simulation algorithm

1. Split $\mathbb{R}$ into disjoint intervals $(A_k)_k$ of the form $A_k = (k, k + 1]$ (length 1 intervals)

2. For each interval $A_k = (k, k + 1]$ we decide how many spikes to put inside the interval by generating $Y_k \sim \text{Pois}(\lambda)$

3. Once we know that $Y_k = m$, we generate $\xi_1, \ldots, \xi_m \sim \text{Unif}(k, k + 1]$ independently
Two Dimensional Poisson Point Process

Denote by \((S_k, Z_k), k \geq 1\) the points on \(\mathbb{R}^2\).
For any subset \(A \cup \mathbb{R}^2\) consider \(M(A)\) the number of points which occurred in \(A\).
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(iii) Given that \(M(A) = k\) the \(k\)-points are uniformly distributed in \(A\).

(iv) \(E[M(A)] = \lambda |A|\) (in average \(\lambda\) spikes per \((r, r+1] \times (r', r'+1]\) square)
Simulation algorithm

1. Split $\mathbb{R}^2$ into disjoint squares $(A_k)_k$ of the form $A_k = (k, k + 1] \times (k', k' + 1]$ (size 1 squares)

2. For each square $A_k$ we decide how many spikes to put inside the square by generating $Y_k \sim \text{Pois} (\lambda)$

3. Once we know that $Y_k = m$, we generate $\xi_1, \ldots, \xi_m \sim \text{Unif} (A_k)$ independently
Let $M$ be a two dimensional Poisson process of rate 1. For each interval $I \cup \mathbb{R}$

$$N(I) = M(I \times [0, \lambda]).$$

**Proposition:** $N$ is a one-dimensional Poisson process of rate $\lambda$. 
One dimensional homogeneous Poisson process
One dimensional homogeneous Poisson process
One dimensional homogeneous Poisson process
One dimensional non-homogeneous Poisson process
One dimensional non-homogeneous Poisson process
One dimensional non-homogeneous Poisson process
Problem: If we do not have the $\times$’s? Can we generate them?
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Answer: it depends.
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We have considered the acknowledgment of a PPP(1) on the entire $R^2$!
**Problem:** If we do not have the $\times$’s? Can we generate them?

Answer: it depends.

We have considered the acknowledgment of a PPP(1) on the entire $R^2$! Can we restrict the $\times$’s we need into a finite space?
Uniformly bounded rate
System of interacting neurons
Biological Motivation

- Each neuron spikes with probability depending on its membrane potential
Biological Motivation

- Each neuron spikes with probability depending on its membrane potential
- **Chemical synapses**
Each neuron spikes with probability depending on its membrane potential.

**Chemical synapses**

- If a neuron $i$ spikes at time $t$
  - its potential is reset to 0 (resting state)
    \[ V_i(t) = 0 \]
  - the membrane potential of other neurons receive a value $w_{i\rightarrow j} \in \mathbb{R}$
    \[ \forall j \neq i, V_j(t) = V_j(t^-) + w_{i\rightarrow j} \] (synaptic weight)
Evolution

\[ \begin{align*}
&k \quad v_k \\
&j \quad v_j \\
&i \quad v_i
\end{align*} \]

\[ k \rightarrow j \rightarrow i \rightarrow t \]
Evolution

\[ v_j + w_{k\rightarrow j} \]

\[ v_i + w_{k\rightarrow i} \]

\[ k \rightarrow j \rightarrow i \rightarrow t \]

\[ T_1 \rightarrow 0 \]
Evolution

\[ v_j + w_{k \rightarrow j} \]

\[ v_i + w_{k \rightarrow i} \]

\[ T_1 \]

\[ t \]
Evolution

\[
I^{T_1} + \omega^j \rightarrow k + \omega^k \rightarrow j + \omega^i \rightarrow j \rightarrow i
\]

\[
I^{T_2} + \omega^i \rightarrow k + \omega^j \rightarrow k + \omega^j \rightarrow i
\]
Evolution

\[ I_{T_1} \rightarrow I_{T_2} \]

\[ w_{i \rightarrow k} \]

\[ v_j + w_{k \rightarrow j} + w_{i \rightarrow j} \]

\[ w_{i \rightarrow j} \]

\[ w_{j \rightarrow i} \]
Evolution

\[ I \xrightarrow{T_1} j \xrightarrow{T_2} k \xrightarrow{T_3} j \]

\[ v_j + w_{k \rightarrow j} + w_{i \rightarrow j} \]

\[ w_{i \rightarrow k} \]
Evolution

\[ w_{i \rightarrow k} + w_{j \rightarrow k} \]

\[ w_{j \rightarrow i} \]
Evolution

\[ T_1, T_2, T_3 \]

\[ w_{i \rightarrow k} + w_{j \rightarrow k} \]

\[ w_{j \rightarrow i} \]

\[ w_{i \rightarrow j} \]
Evolution
Evolution

\[
\begin{align*}
I & \rightarrow T_1 \\
T_1 & \rightarrow T_2 \\
T_2 & \rightarrow T_3 \\
T_3 & \rightarrow T_4
\end{align*}
\]

\[
2w_{i \rightarrow k} + W_{j \rightarrow k}
\]

\[
w_{i \rightarrow j}
\]

\[0\]
Evolution

$\begin{align*}
T_1 & : w_{i \rightarrow k} + W_{j \rightarrow k} \\
T_2 & : w_{i \rightarrow j} \\
T_3 & : w_{i \rightarrow j} \\
T_4 & : 0
\end{align*}$
▶ **Leak channels:** between consecutive spikes the membrane potential of each neuron loses potential *continuously* to the environment.

- Suppose that at time $t_0$, it holds $V_i(t_0) = v$.
- For any $t > t_0$ such that there was not spikes in $[t_0, t]$

\[
V_i(t) = ve^{-\alpha(t-t_0)}
\]
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that is, for any \( s \in [t_0, t] \), \( V(t) \) satisfies the ODE

\[
\frac{dV(t)}{dt} = -\alpha V(t) \quad (V_{rest} = 0)
\]
**Biological Motivation**

▶ **Leak channels:** between consecutive spikes the membrane potential of each neuron loses potential *continuously* to the environment.

▷ Suppose that at time $t_0$, it holds $V_i(t_0) = v$.
▷ For any $t > t_0$ such that there was not spikes in $[t_0, t]$

\[ V_i(t) = ve^{-\alpha(t-t_0)} \]

that is, for any $s \in [t_0, t]$, $V(t)$ satisfies the ODE

\[ \frac{dV(t)}{dt} = -\alpha V(t) \quad (V_{\text{rest}} = 0) \]

▷ If at the time $s > t > t_0$ neuron $j$ spikes, $\forall i \neq j$,

\[ V_i(t) = ve^{-\alpha(s-t_0)} + w_{j \rightarrow i}, \quad \text{and} \quad V_j(t) = 0 \]
Imagine a nice figure here

Sorry, I wasn’t able to do it in LateX :/
$T_k$ denotes the $k$-spike of the system
$T_k$ denotes the $k$-spike of the system
Suppose
- $1o$ spike was duo to neuron $k$
- $2o$ spike was duo to neuron $\ell$
- $3o$ spike was duo to neuron $i$
Can we simulate the system using the $\times$’s?
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The spiking rates are clearly nonhomogeneous.
Can we simulate the system using the $\times$’s?

The spiking rates are clearly nonhomogeneous. Are the spiking rates uniformly upper bounded?
Can we simulate the system using the \( \times \)'s?

The spiking rates are clearly nonhomogeneous. Are the spiking rates \textbf{uniformly} upper bounded? What happens with the spiking rate of a neuron, when another neuron spikes?
Can we simulate the system using the $\times$’s?

The spiking rates are clearly nonhomogeneous.

Are the spiking rates **uniformly** upper bounded?

What happens with the spiking rate of a neuron, when another neuron spikes? Can we deal with this?
How to deal with two Poisson process simultaneously?
How to deal with two Poisson process simultaneously?
How to deal with two Poisson process simultaneously?
How to define $T_{k+1} - T_k$ for the system?

Note

$$T_1 = \min\{T^\lambda_1, T'^\lambda_1\}$$
How to define $T_{k+1} - T_k$ for the system?

Note

$$T_1 = \min\{T_1^\lambda, T_1^{\lambda'}\}$$

**Proposition:** If $X \sim E(\lambda)$ and $Y \sim E(\lambda')$ are independent. Then

$$\min\{X, Y\} \sim E(\lambda + \lambda')$$
Simulation Algorithm

1. Generate a $T_{k+1} - T_k \sim \mathcal{E}(\lambda + \lambda')$
2. Generate a $\xi_k \sim \text{Unif}(0, \lambda + \lambda')$
   2.1 If $0 < \xi \leq \lambda$ then the green neuron spikes
   2.2 If $\lambda < \xi \leq \lambda + \lambda'$ then the red neuron spikes
How about nonhomogeneous Poisson process?
How about nonhomogeneous Poisson process?
How about nonhomogeneous Poisson process?
How about nonhomogeneous Poisson process?
Set $L = 0$

1. Generate a $S \sim \mathcal{E}(M)$

2. Generate a $\xi \sim \text{Unif}(0, M)$
   
   2.1 If $0 < \xi \leq \lambda(S)$ then
       
       $T_1 = S$

       the green neuron spikes at time $T_1$

   2.2 If $\lambda(S) < \xi \leq \lambda(S) + \lambda'(S)$ then
       
       $T_1 = S$

       the red neuron spikes at time $T_1$

   2.3 If $\lambda(S) + \lambda'(S) < \xi < M$ no one spikes and
       
       set $L = S$. 
How about stochastic IF model??

Remember the spiking rate looks like. Can we apply the simulation approach for nonhomogeneous Poisson process? Answer: Yes, but piecewisely!!
How about stochastic IF model??

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Can we apply the simulation approach for nonhomogeneous Poisson process?
How about stochastic IF model??

Remember the spiking rate look like

Can we apply the simulation approach for nonhomogeneous Poisson process?

**Answer:** Yes, but **piecewisely**!!
Simulation Algorithm: consider $\varphi(v) = v$

Given the initial potentials $v_1, \ldots, v_N$. Define $M = \sum v_k$ and set $L = 0$ and $T_0 = 0$.

1. Generate a $S \sim \mathcal{E}(M)$ (number with exponential distribution)
   Update $L = L + S$

2. Generate a $\xi \sim Unif(0, M)$ (number with uniform distribution)

3. Update $v_k = v_k e^{-\alpha S}$
   3.1 If $0 < \xi \leq v_1$ then
      $T_m = T_{m-1} + S$ (accept as a jumping time of the system)
      $V_1(L) = 0$ and
      $V_j(L) = v_j + w_{1 \rightarrow j}$, $j = 2, \ldots, N$ (chemical synapses)
   3.2 If $\sum_1^k v_k < \xi \leq \sum_1^{k+1} v_k$ then
      $T_m = T_{k-m} + S$ (accept as a jumping time of the system)
      $V_k(L) = 0$ and $V_j(L) = v_j + w_{k \rightarrow j}$, $j \neq k$ (chemical synapses)
   3.3 If $\sum_1^N v_k < \xi < M$ then no one spikes
      $V_k(L) = v_k \forall k$

4. Update $M = \sum_1^N V_k(L)$
Simulation Algorithm: consider \( \varphi(v) = v \)

Given the initial potentials \( v_1, \ldots, v_N \). Define \( M = \sum \varphi(v_k) \) and set \( L = 0 \) and \( T_0 = 0 \).

1. Generate a \( S \sim \mathcal{E}(M) \) (number with exponential distribution)
   Update \( L = L + S \)

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   3.1 If \( 0 < \xi \leq \varphi(v_1) \) then
      \( T_m = T_{m-1} + S \) (accept as a jumping time of the system)
      \( V_1(L) = 0 \) and
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   3.2 If \( \sum_1^k \varphi(v_k) < \xi \leq \sum_1^{k+1} \varphi(v_k) \) then
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      \( V_k(L) = 0 \) and \( V_j(L) = v_j + w_{k\rightarrow j}, \ j \neq k \) (chemical synapses)
   3.3 If \( \sum_1^N \varphi(v_k) < \xi < M \) then no one spikes
      \( V_k(L) = v_k \ \forall k \)

4. Update \( M = \sum_1^N \varphi(V_k(L)) \)
Obrigada