Stochastic models in neuroscience:

Poisson point process simulation approach

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LASCON VIII

Membrane potential and spikes

Membrane potential	=	**difference** in voltage between the inner and outer cellular membrane

Membrane potential and spikes

$$\text{Membrane potential} = V(t) \in \mathbb{R}$$

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Temporal evolution

1. A neuron accumulate potential duo to a stimuli current $I(t)$
   (interaction with other neurons ou external stimuli)
2. The occurrence of a spike in a neuron depends on its membrane potential (the higher the potential the higher the spiking probability)
3. When a neuron spikes, it loses all the accumulated potential

Single integrate-and-fire model

Deterministic version: the membrane potential

- integrate a current up to a threshold
- "spike"
- reset the membrane potential to a resting value

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Formally

Let $C(t)$ be a stimuli current

Deterministic version:

Fix a threshold ν and denote by $L = sup\{s < t : V(s) = V_{rest}\}$, then

$$V(t) = \int_L^t C(s)ds \quad \text{up to a threshold } \nu.$$

If $V(t^-) = \nu$, then set $V(t) = V_{rest}$.

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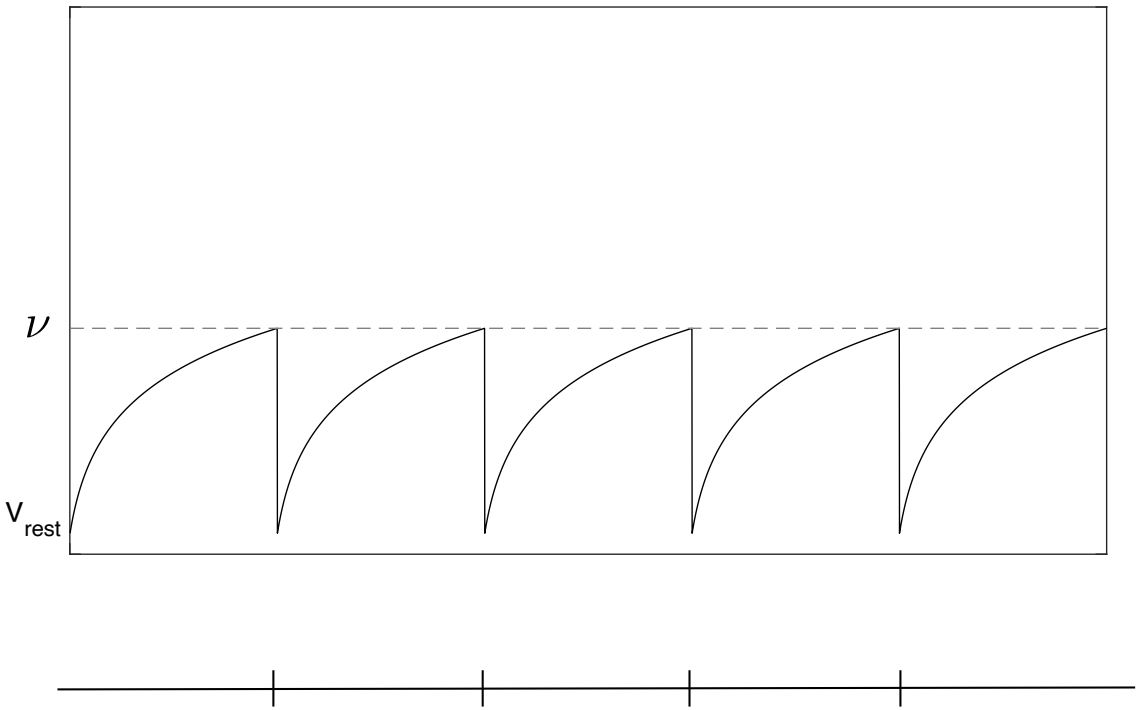

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Then

$$P\big(V(s) = V_{rest} \ for \ some \ s \in [t, t + dt] \mid V(t)\big)$$

depends on $V(t)dt$

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Then

$$P\big(V(s) = V_{rest} \ for \ some \ s \in [t, t + dt] \mid V(t)\big) \approx \varphi(V(t))dt$$

with $\varphi : \mathbb{R} \to [0, 1]$ a given spiking rate function

How can we do it?

Poisson Point Process

One Dimensional Poisson Point Process

Denote by $T_k, k \geq 1$ the spiking times of the neuron

For any interval $A \cup \mathbb{R}$ consider $N(A)$ the number of spikes which occurred in A.

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(i) For each interval $A \cup \mathbb{R}$, $N(A)$ has Poisson distribution

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(v) $E[N(A)] = \lambda|A|$ (in average λ spikes per $(r, r+1]$ interval)

Simulation algorithm

1. Split $\mathbb{R}$ into disjoint intervals $(A_k)_k$ of the form $A_k = (k, k+1]$ (length 1 intervals)
2. For each interval $A_k = (k, k+1]$ we decide how many spikes to put inside the interval by generating $Y_k \sim Pois(\lambda)$
3. Once we know that $Y_k = m$, we generate $\xi_1, \ldots, \xi_m \sim Unif(k, k+1]$ independently

Two Dimensional Poisson Point Process

Denote by $(S_k, Z_k), k \geq 1$ the points on $\mathbb{R}^2$

For any subset $A \cup \mathbb{R}^2$ consider $M(A)$ the number of points which occurred in A.

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(iv) $E[M(A)] = \lambda|A|$ (in average λ spikes per $(r, r+1] \times (r', r'+1]$ square)

Simulation algorithm

1. Split $\mathbb{R}^2$ into disjoint squares $(A_k)_k$ of the form $A_k = (k, k+1] \times (k', k'+1]$ (size 1 squares)
2. For each square A_k we decide how many spikes to put inside the square by generating $Y_k \sim Pois(\lambda)$
3. Once we know that $Y_k = m$, we generate $\xi_1, \ldots, \xi_m \sim Unif(A_k)$ independently

Projection of Poisson process

Let M be a two dimensional Poisson process of rate 1.

For each interval $I \cup \mathbb{R}$

$$N(I) = M(I \times [0, \lambda]).$$

Proposition: N is a one-dimensional Poisson process of rate λ.

One dimensional homogeneous Poisson process

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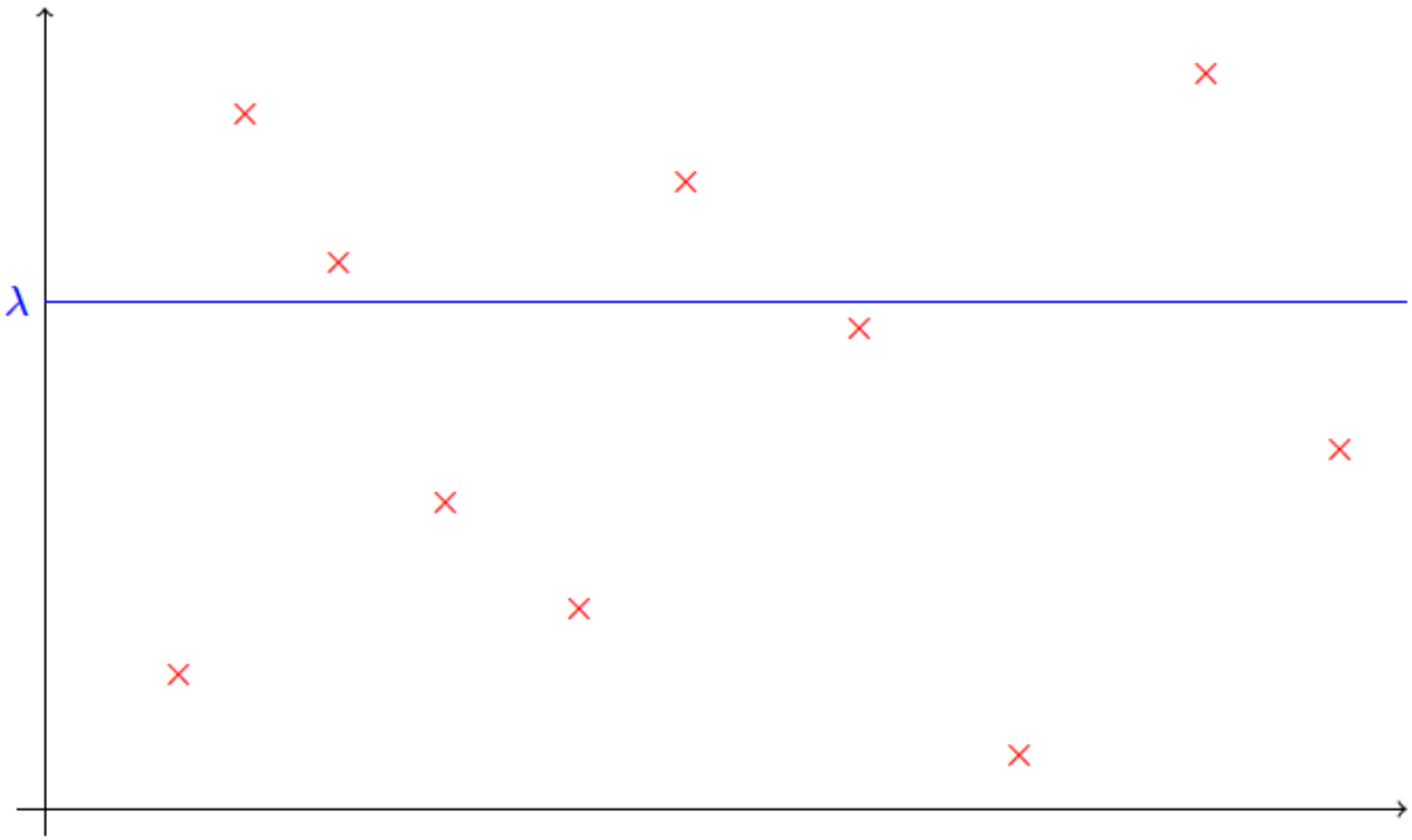

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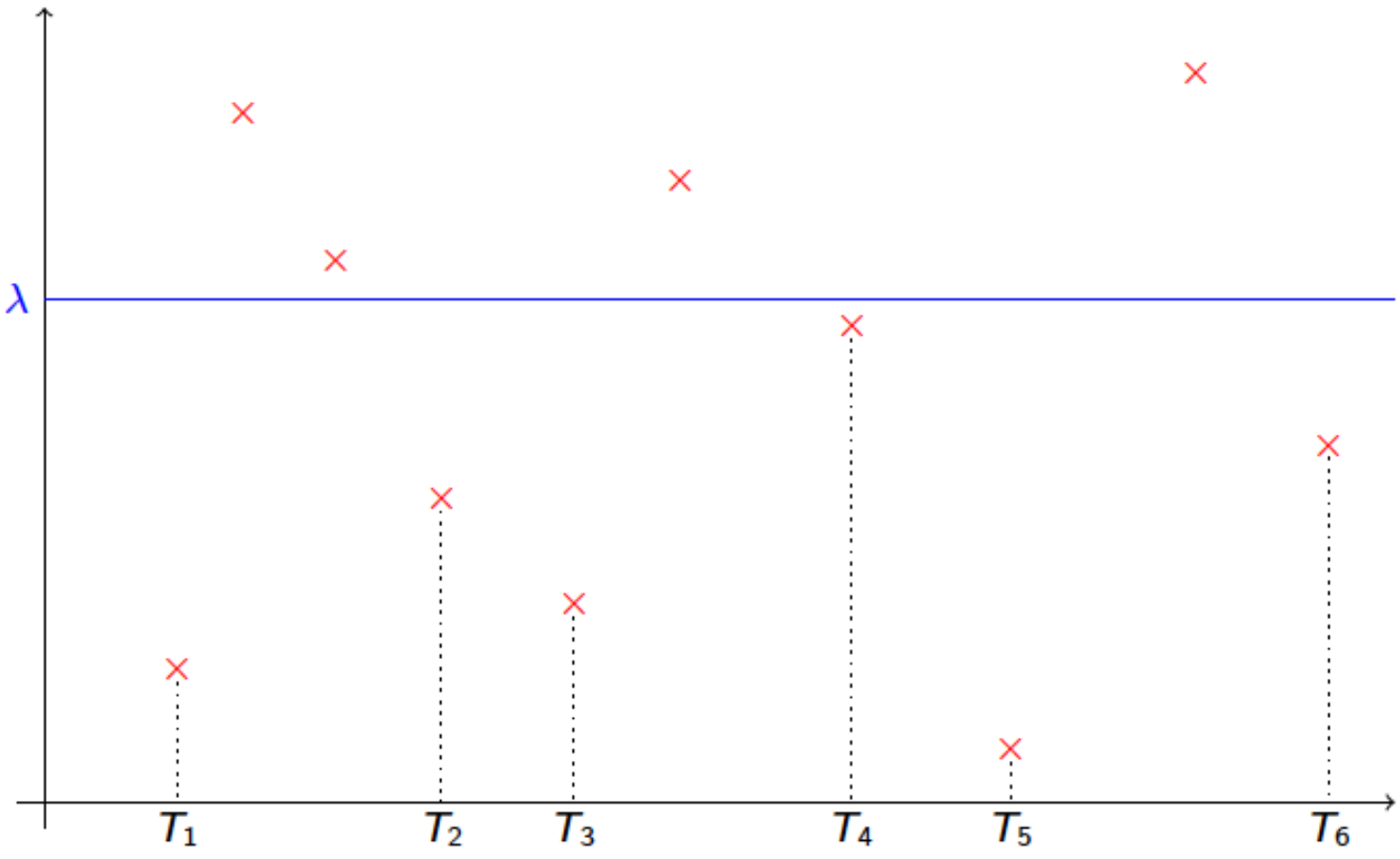

One dimensional non-homogeneous Poisson process

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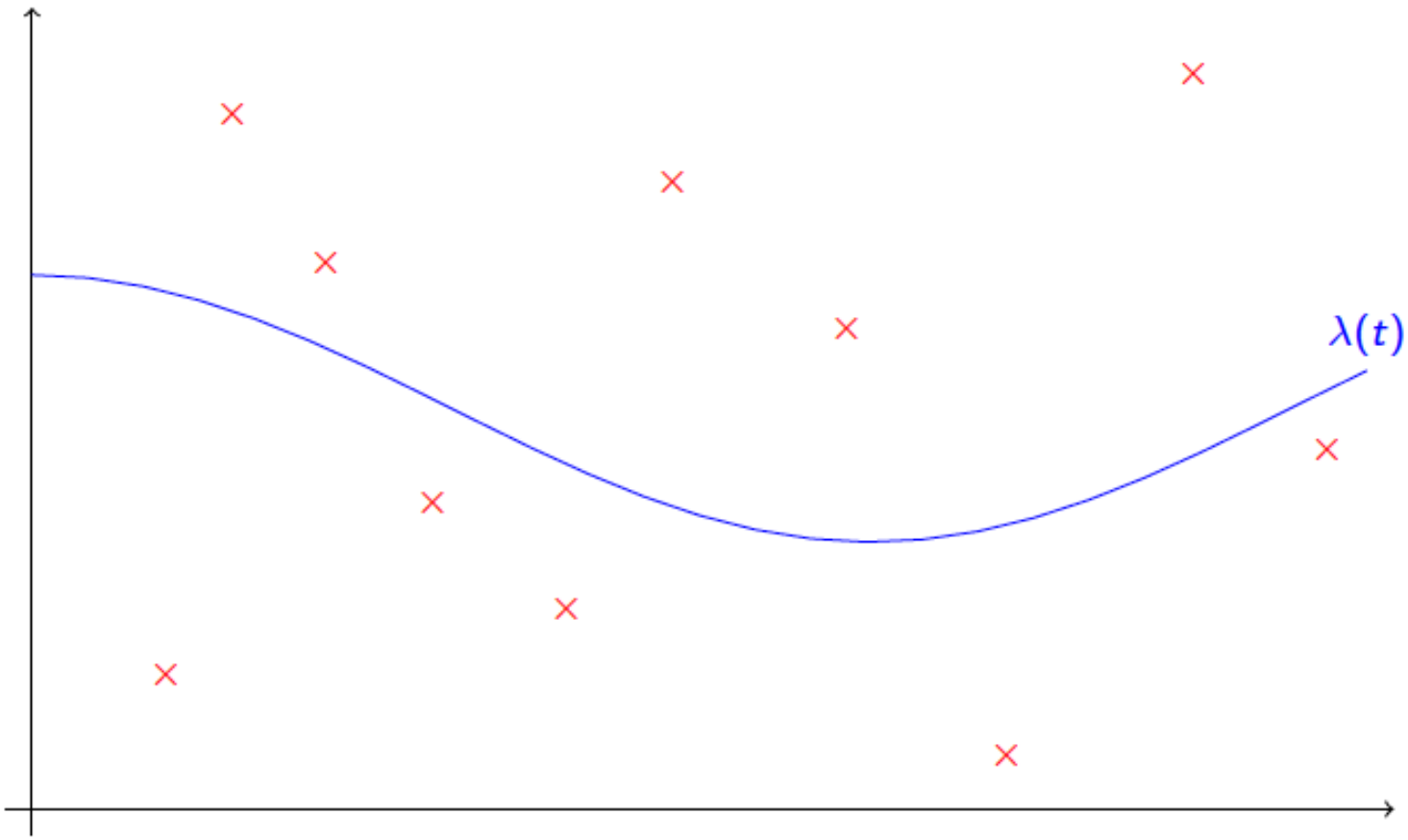

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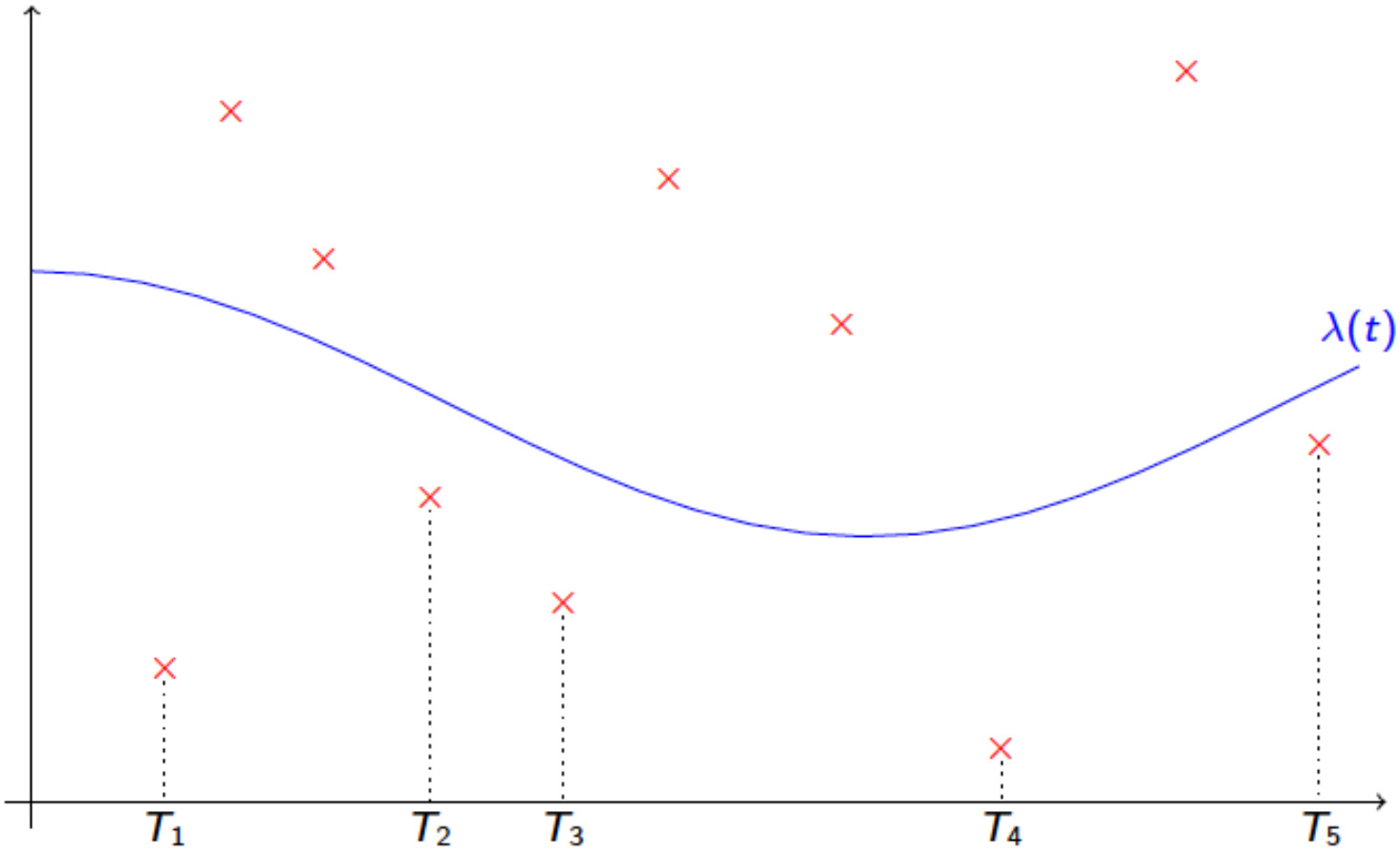

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Can we restrict the $\times$'s we need into a finite space?

Uniformly bounded rate

System of interacting neurons

Biological Motivation

- Each neuron spikes with probability depending on its membrane potential

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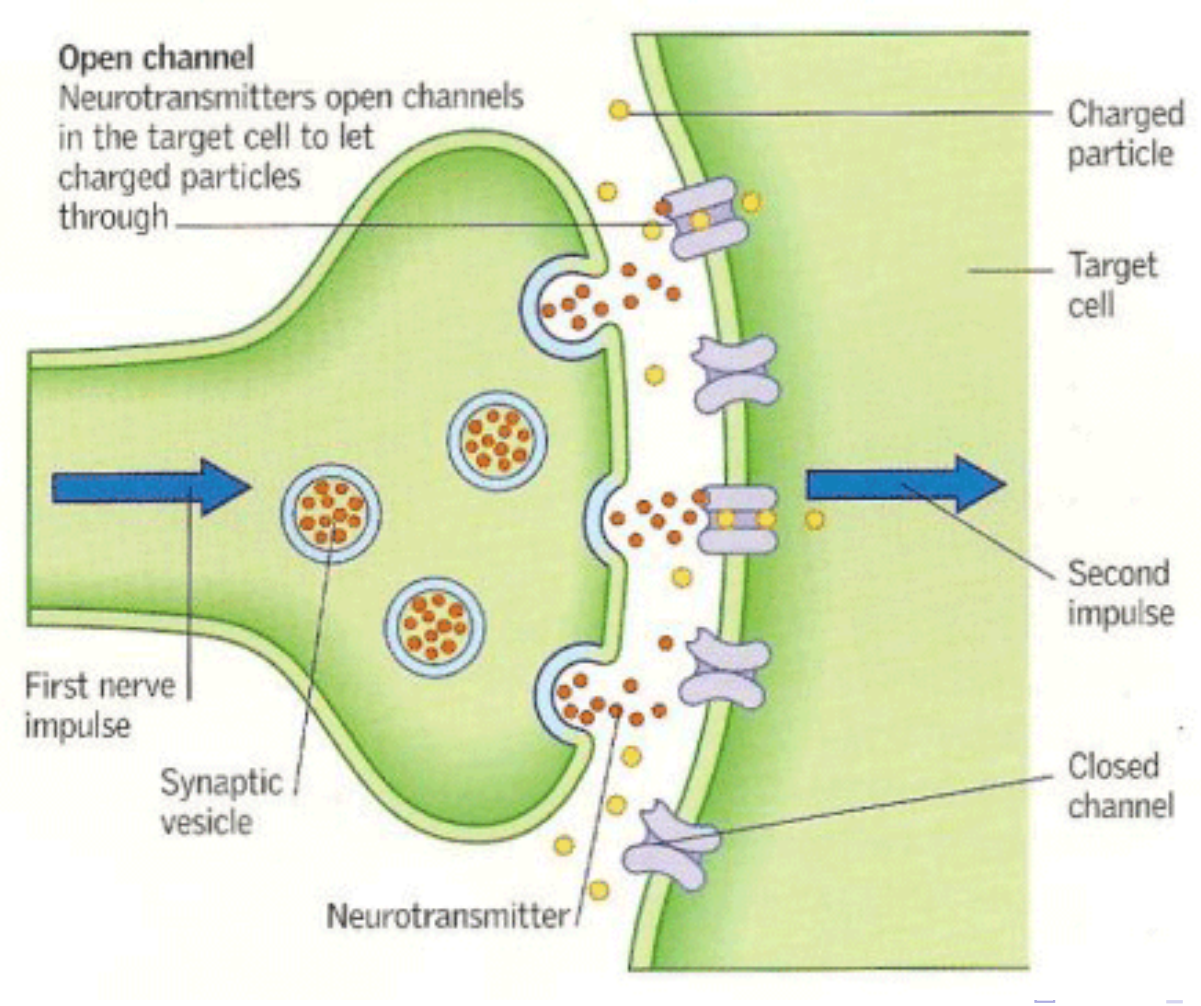

Biological Motivation

- Each neuron spikes with probability depending on its membrane potential
- **Chemical synapses**
  - If a neuron i spikes at time t
    - its potential is **reset** to 0 (resting state)
      $\Rightarrow V_i(t) = 0$
    - the membrane potential of other neurons **receive a value** $w_{i \to j} \in \mathbb{R}$
      $\Rightarrow \forall j \neq i, V_j(t) = V_j(t^-) + w_{i \to j}$ (synaptic weight)

Evolution

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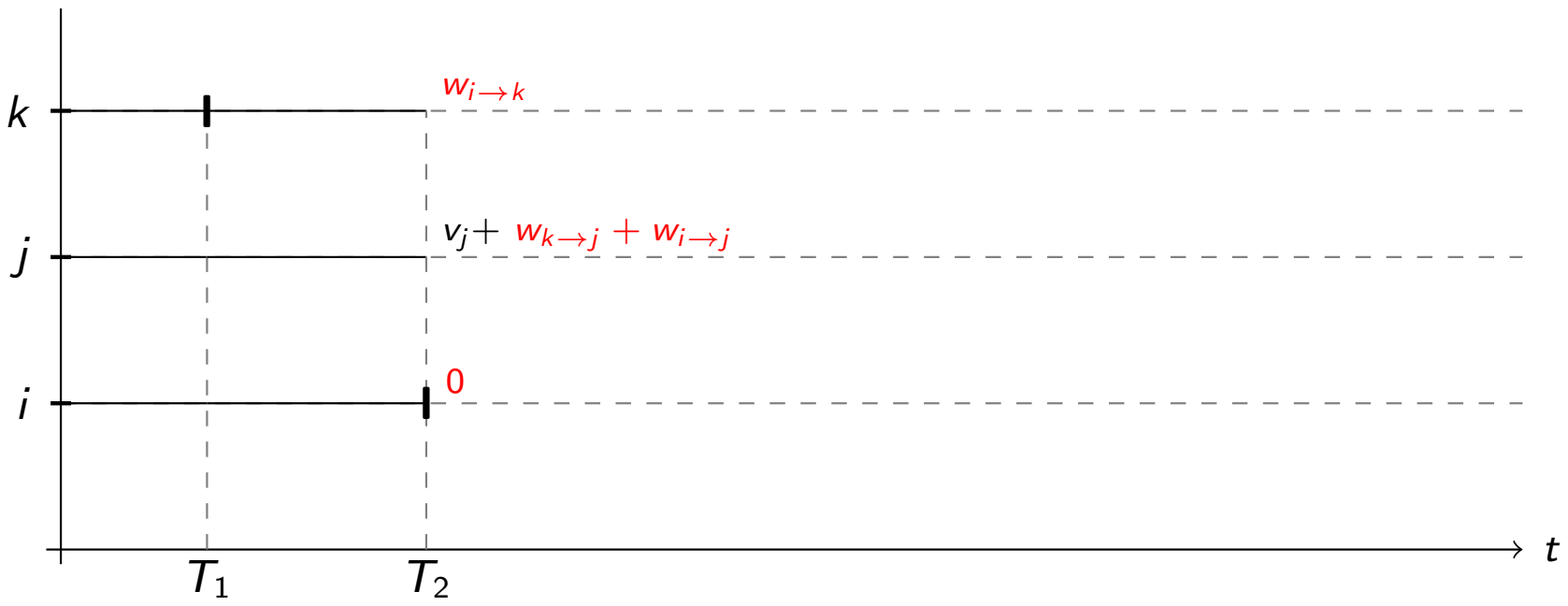

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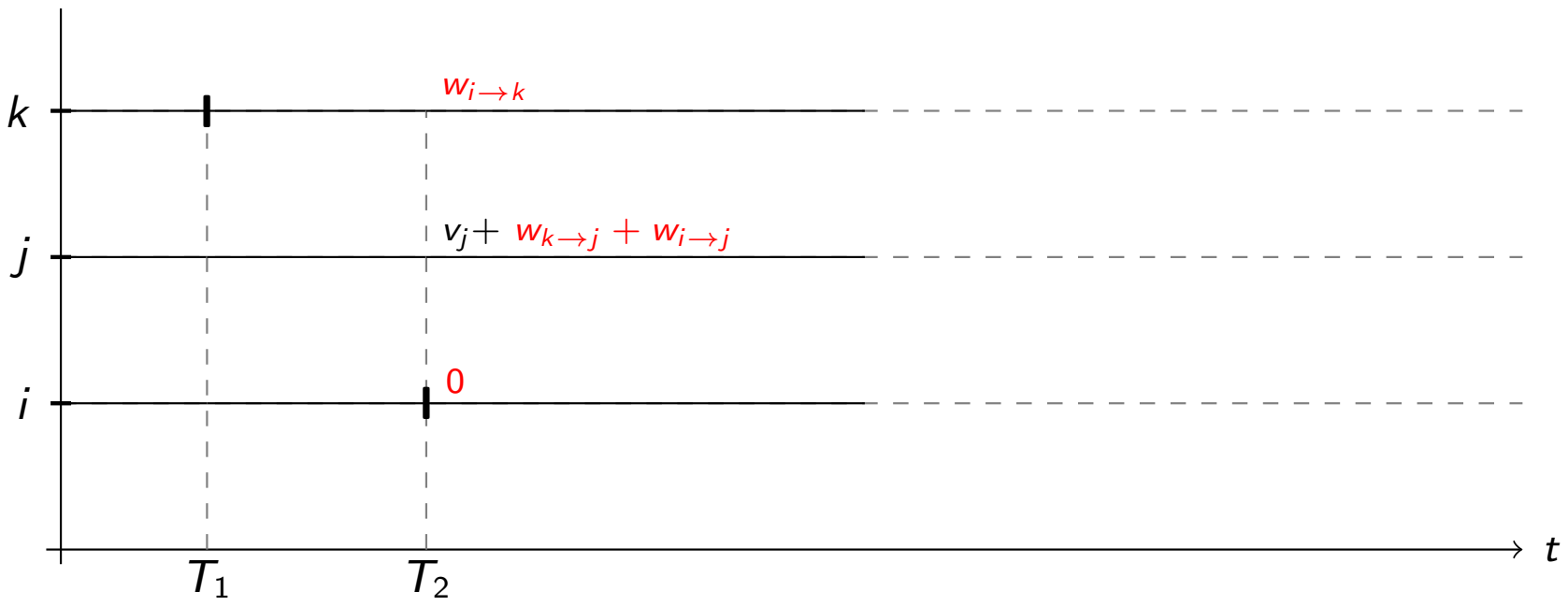

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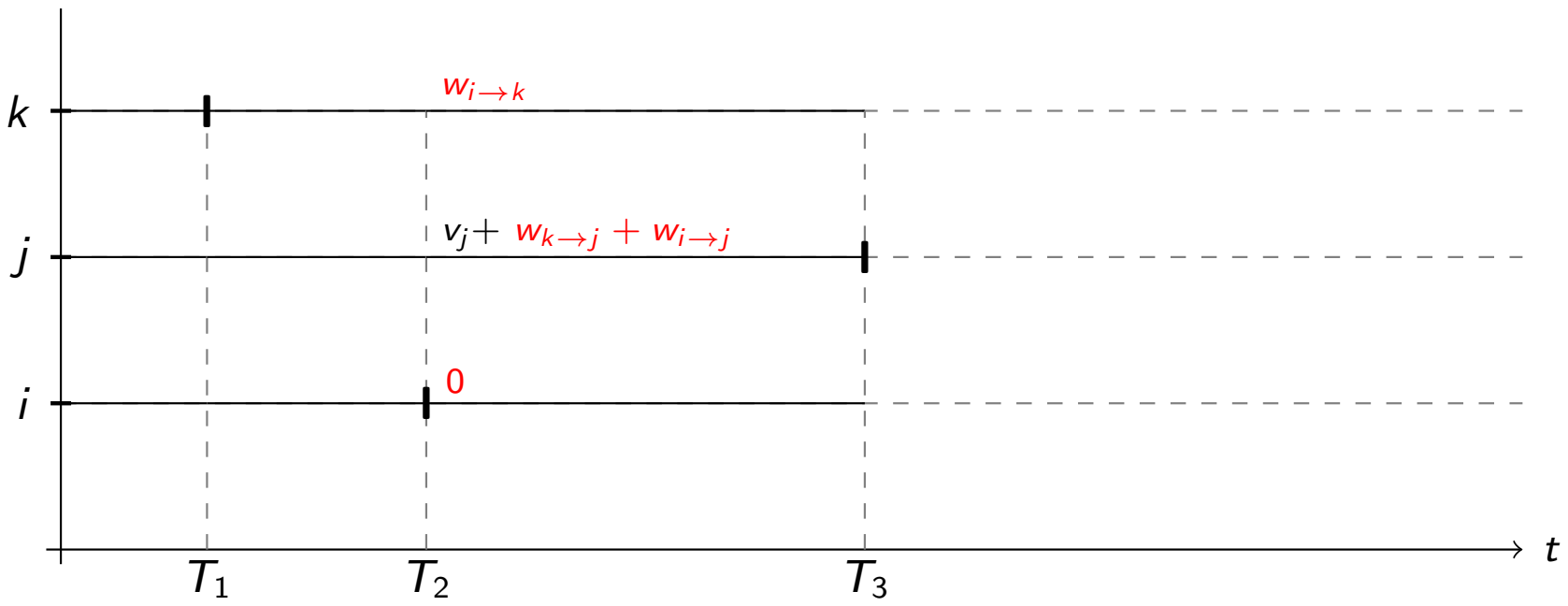

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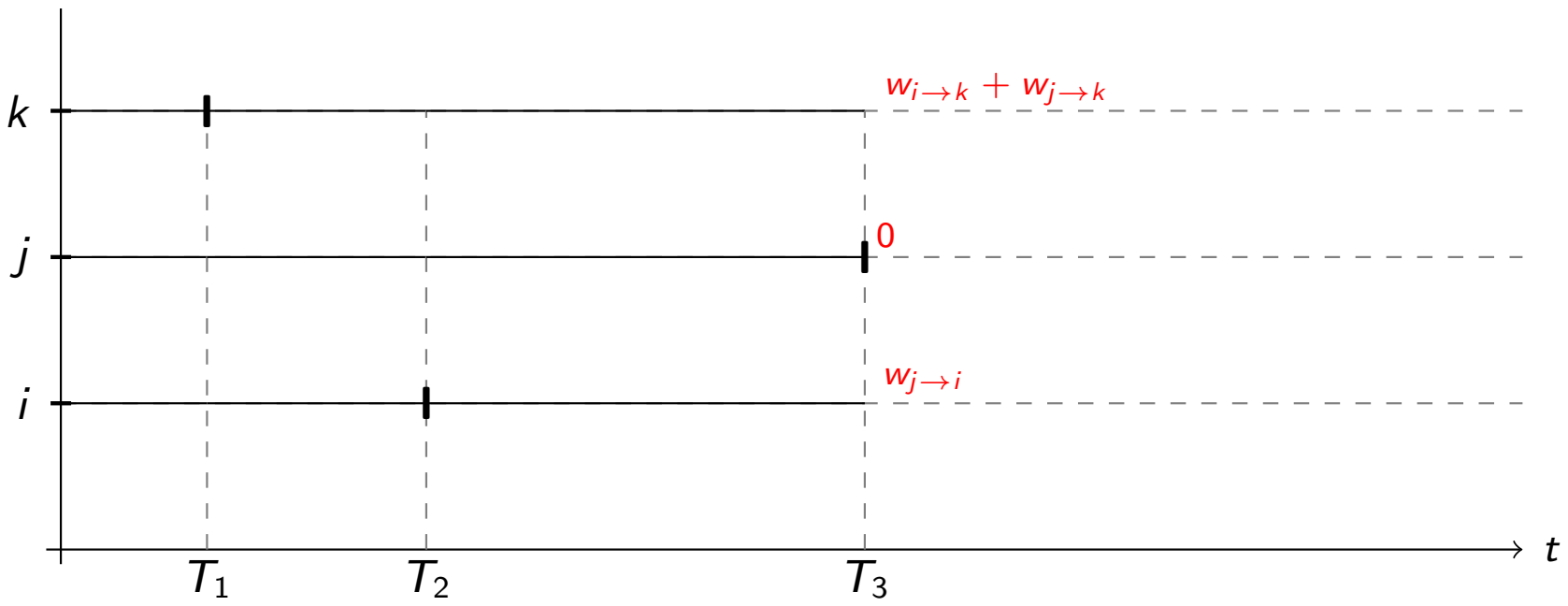

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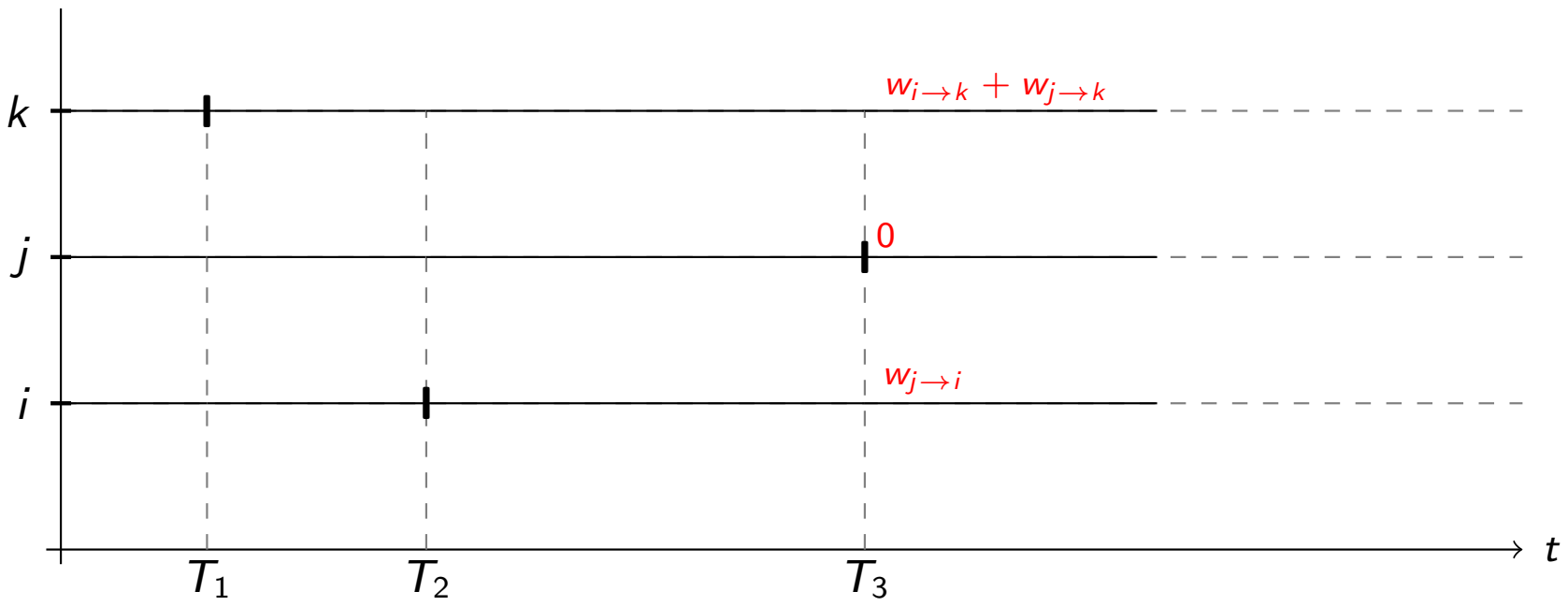

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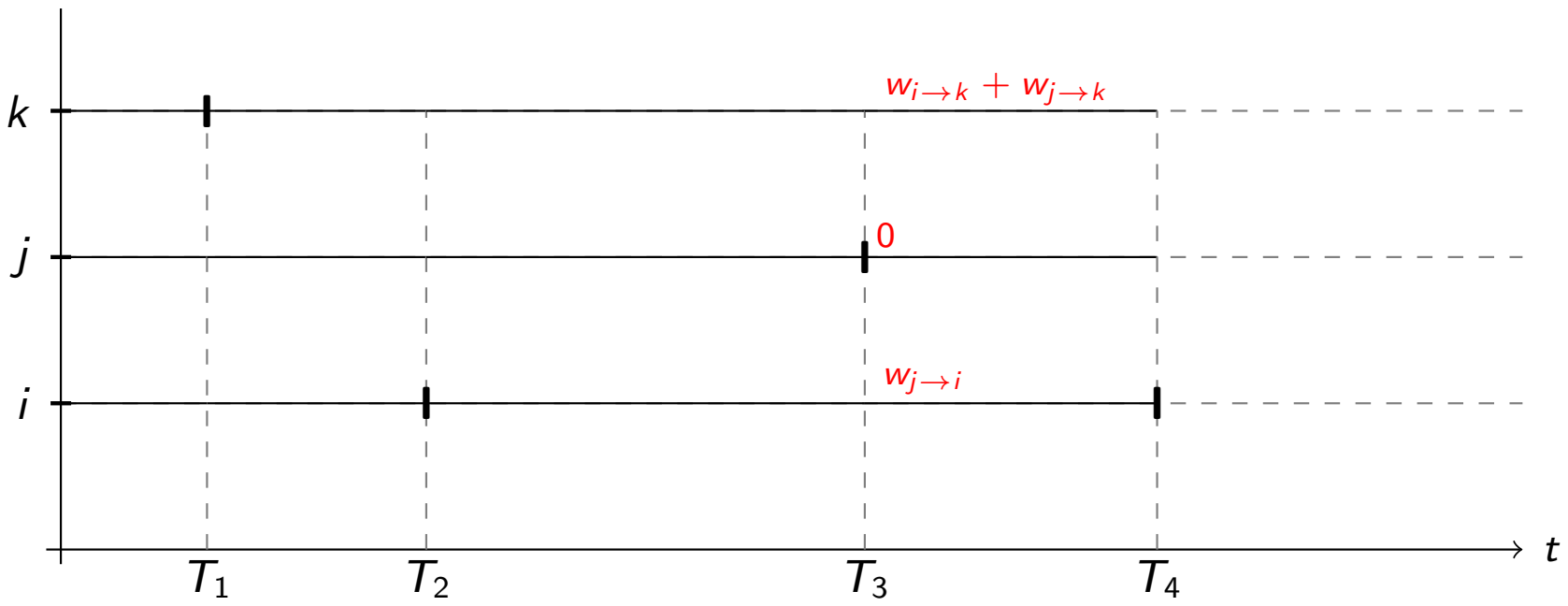

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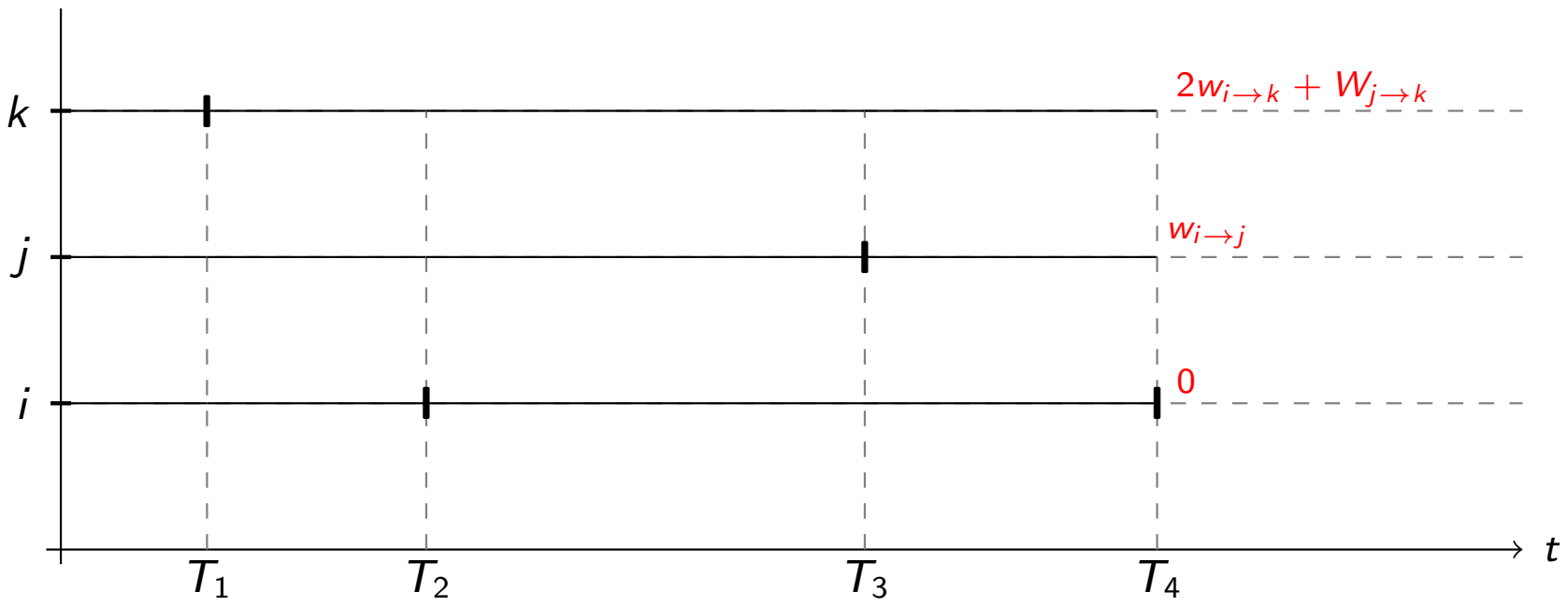

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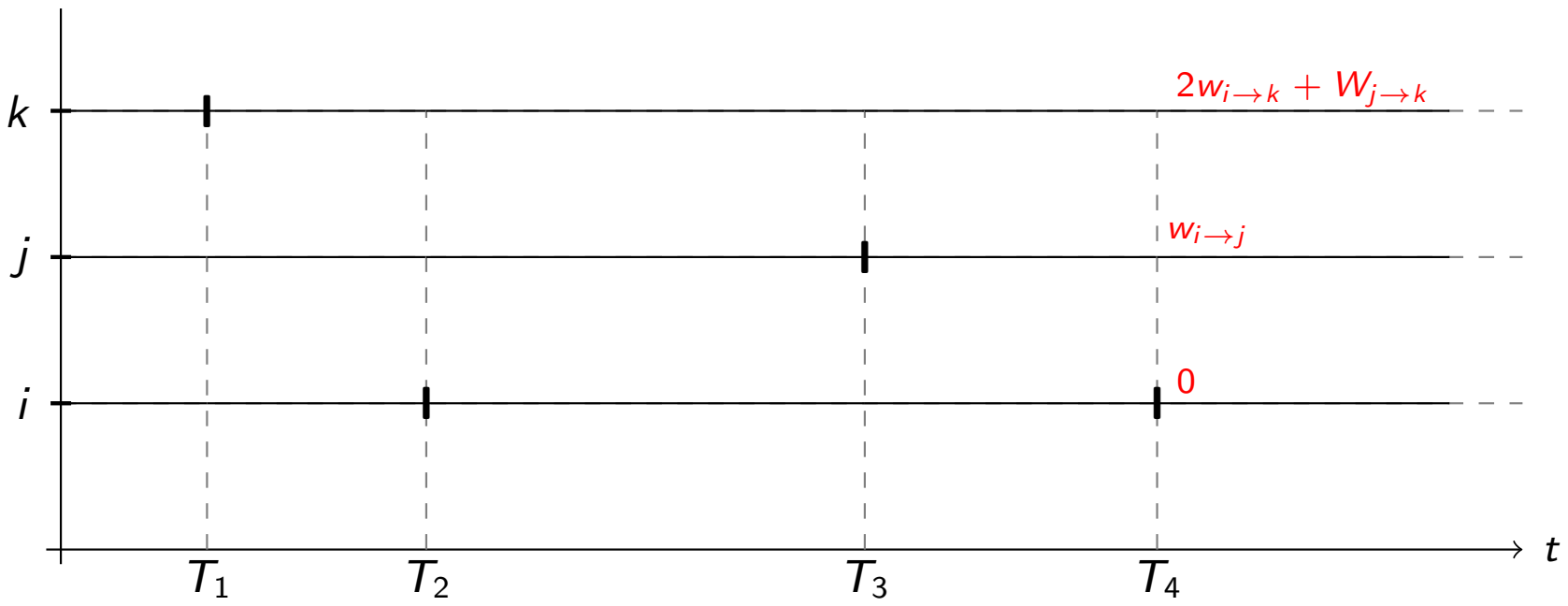

Biological Motivation

- **Leak channels:** between consecutive spikes the membrane potential of each neuron loses potential continuously to the environment.
  - Suppose that at time t_0, it holds $V_i(t_0) = v$.
  - For any $t > t_0$ such that there was not spikes in $[t_0, t]$

$$V_i(t) = ve^{-\alpha(t-t_0)}$$

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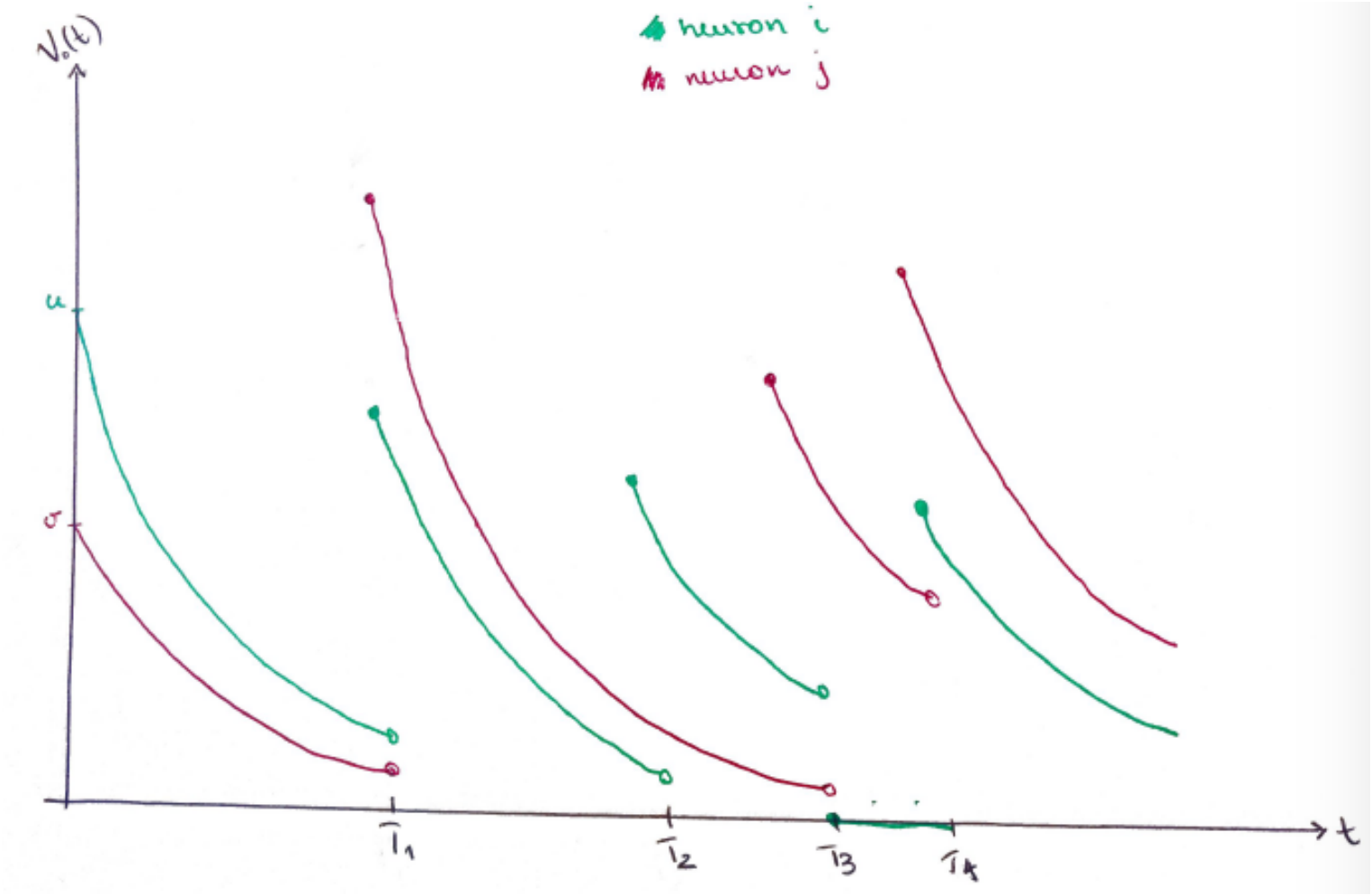
  - If at the time $s > t > t_0$ neuron j spikes, $\forall i \neq j$,

$$V_i(t) = ve^{-\alpha(s-t_0)} + w_{j\to i}, \quad \textit{and} \quad V_j(t) = 0$$

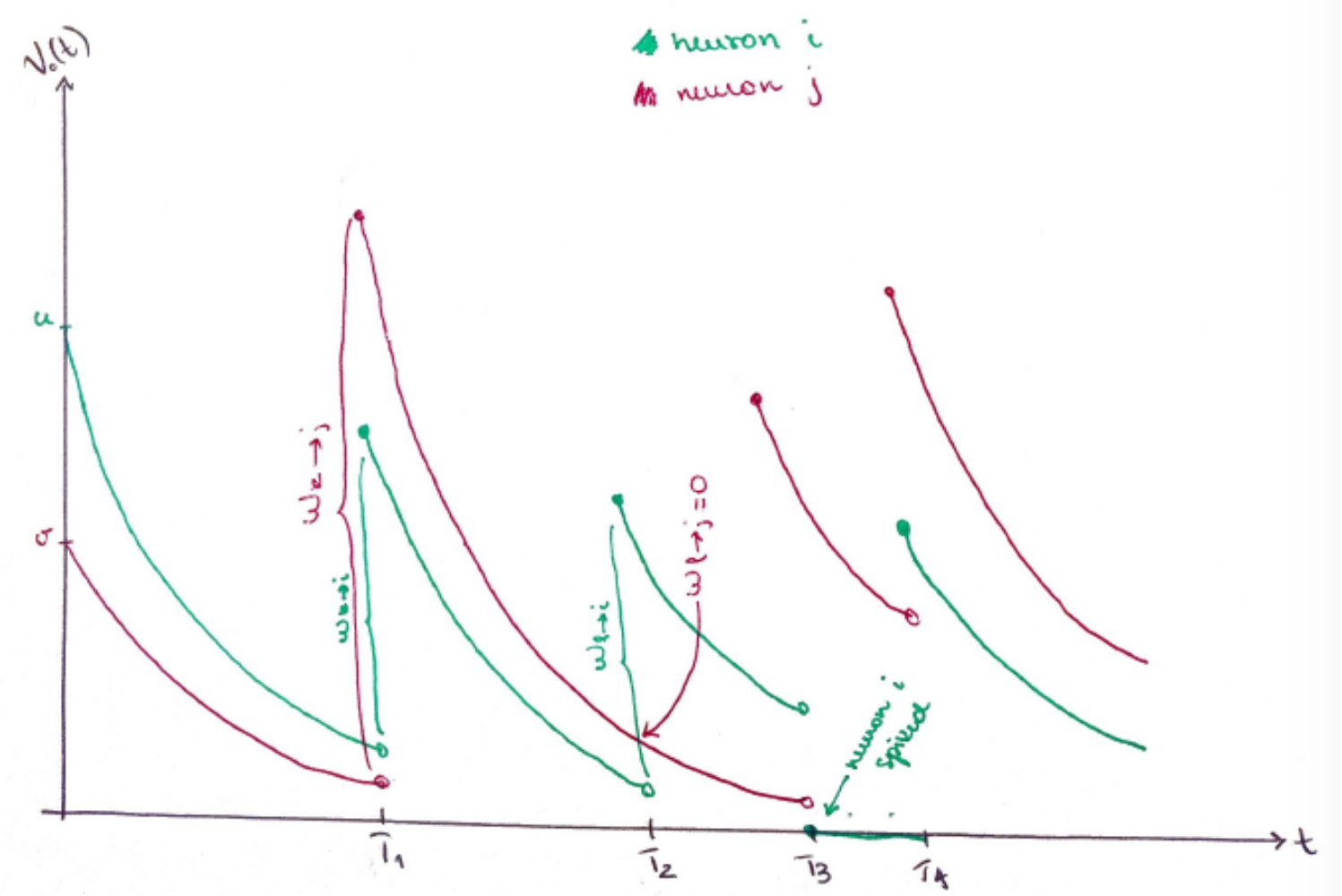
Time evolution illustration

Imagine a nice figure here

Sorry, I wasn't able to do it in LateX :/



T_k denotes the k-spike of the system



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Suppose

- 1o spike was duo to neuron k
- 2o spike was duo to neuron ℓ
- 3o spike was duo to neuron i

Simulation algorithm

Can we simulate the system using the $\times$'s?

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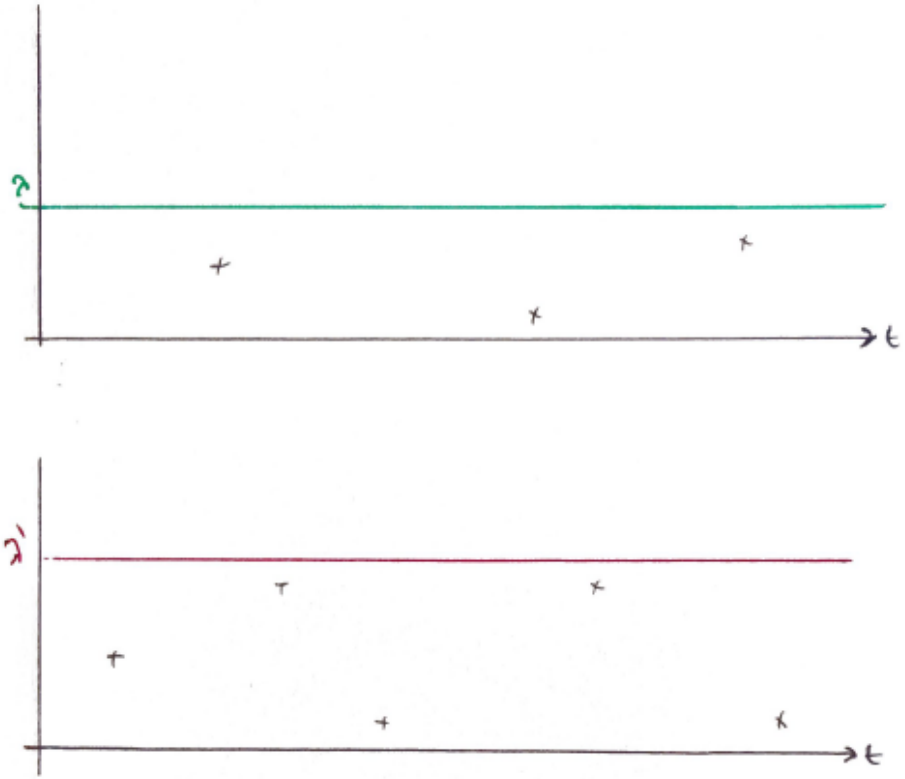
How to deal with two Poisson process simultaneously?



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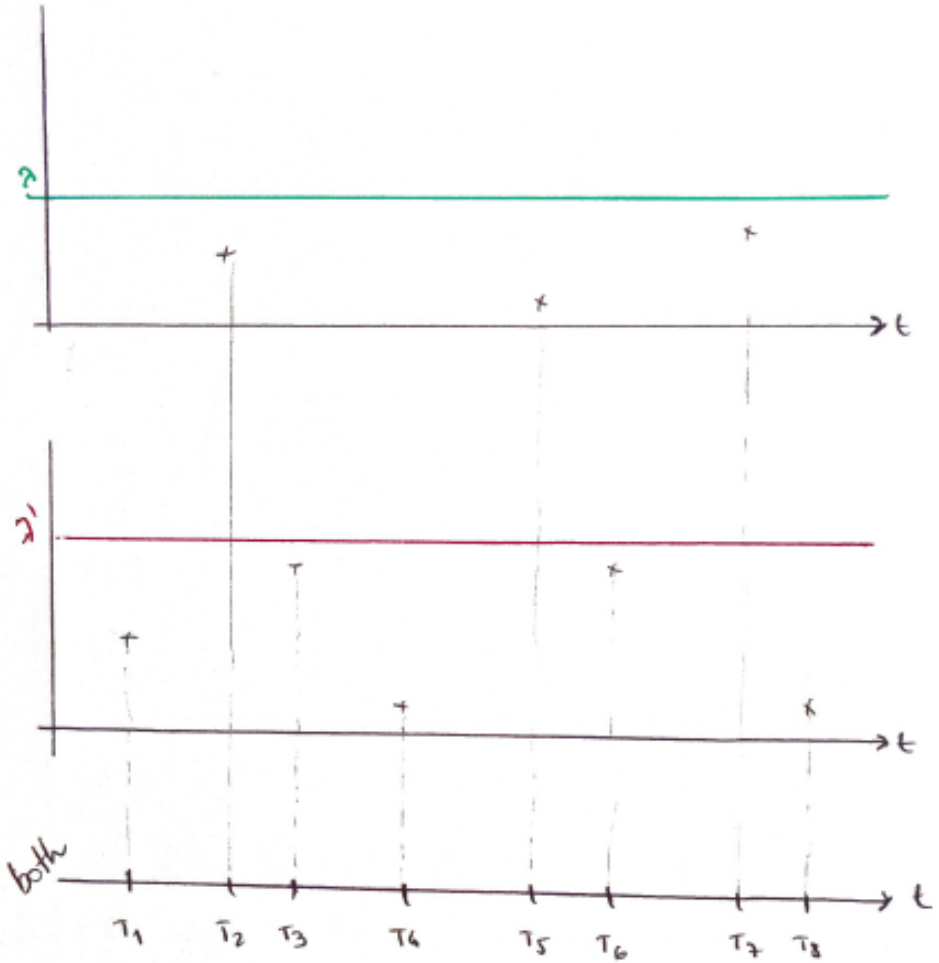

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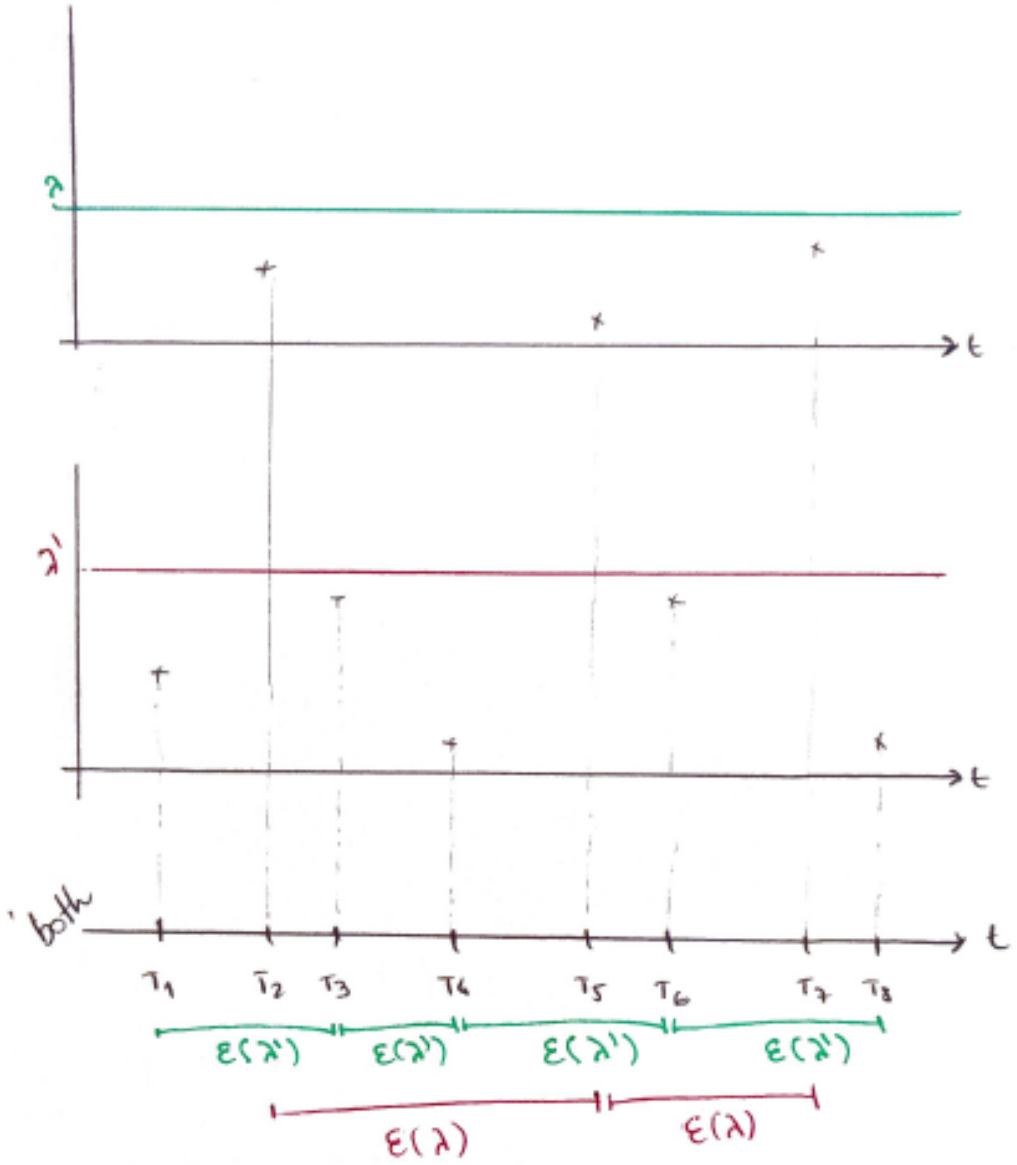

How to define $T_{k+1} - T_k$ for the system?

Note

$$T_1 = \min\{ T_1^{\lambda}, T_1^{\lambda'} \}$$

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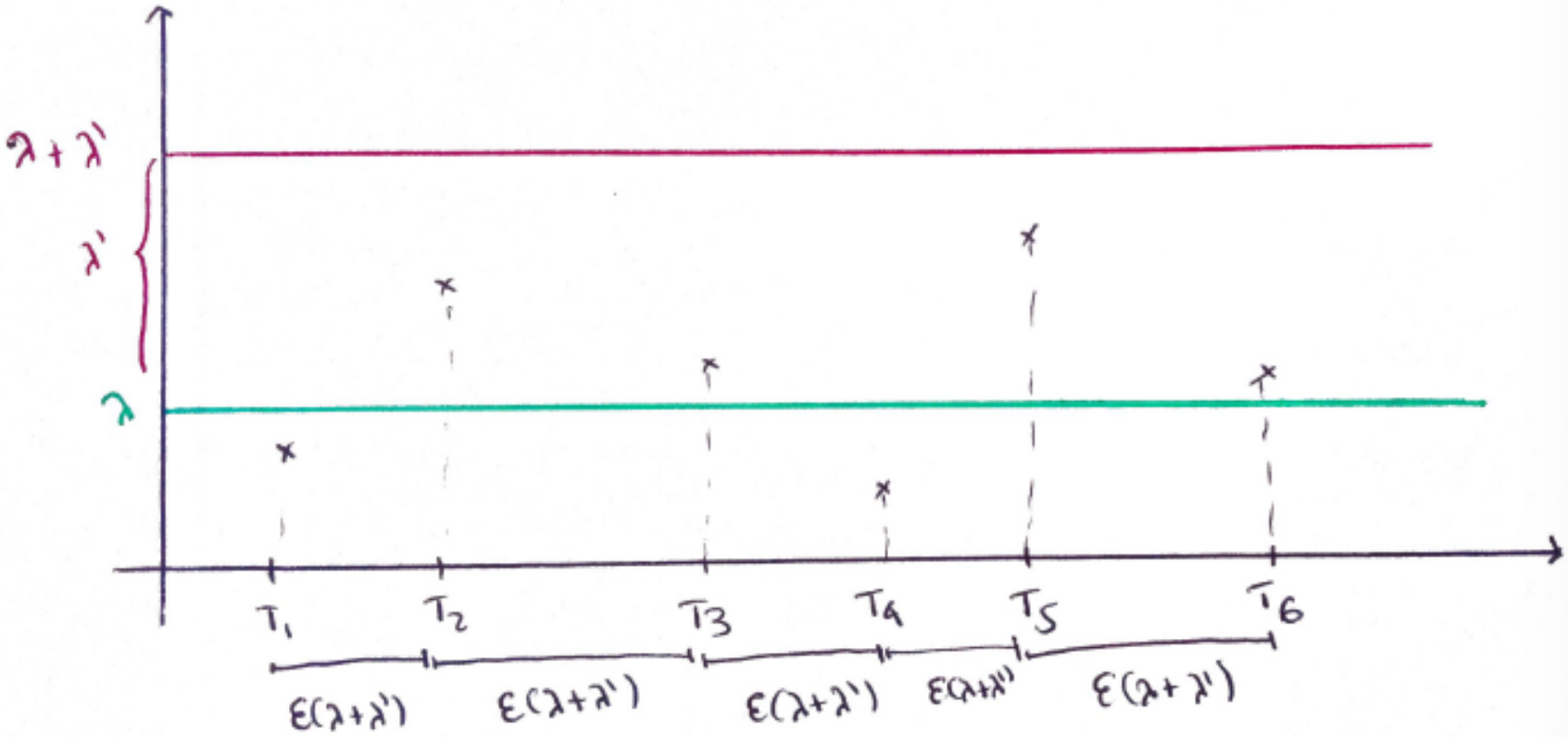
Proposition: If $X \sim \mathcal{E}(\lambda)$ and $Y \sim \mathcal{E}(\lambda')$ are independent. Then $\min\{X, Y\} \sim \mathcal{E}(\lambda + \lambda')$


λ + λ'
λ'
λ
T1
T2
T3
T4
T5
T6
ε(λ+λ')
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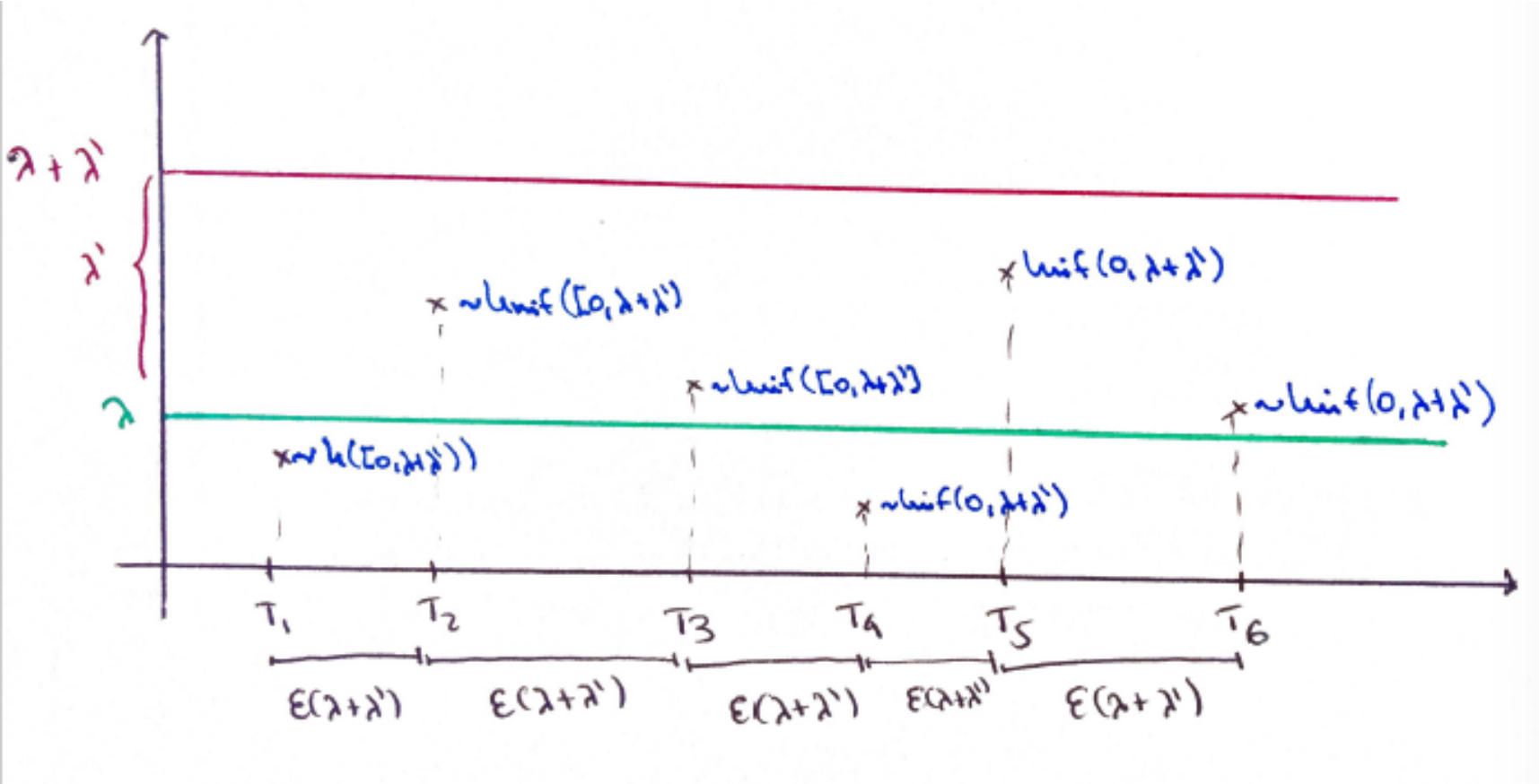
λ + λ'
λ'
λ
~unif([0,λ+λ'])
~unif([0,λ+λ'])
unif(0,λ+λ')
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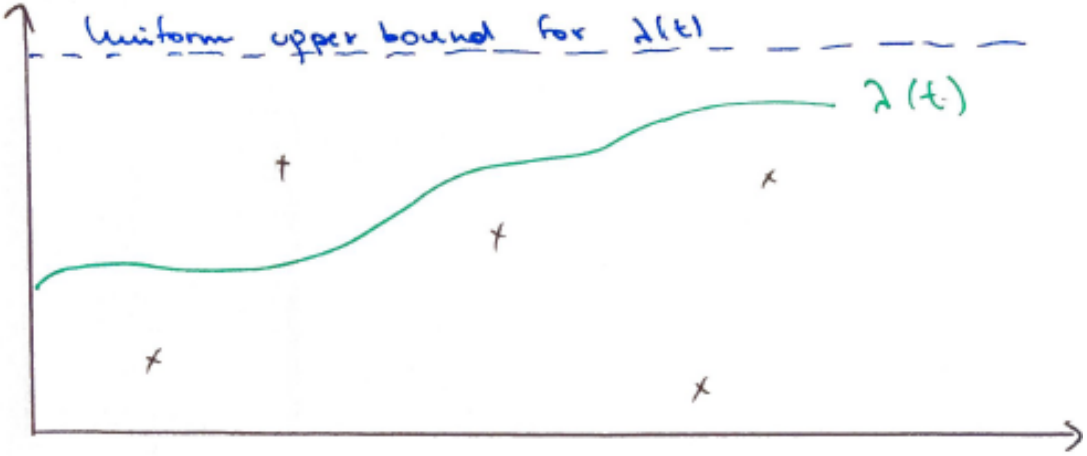
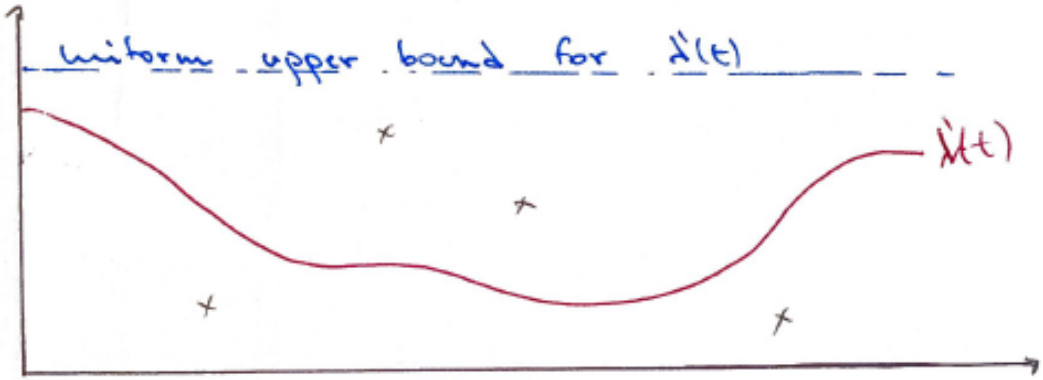
~unif(0,λ+λ')
~unif([0,λ+λ'])
T1
T2
T3
T4
T5
T6
E(λ+λ')
E(λ+λ')
E(λ+λ')
E(λ+λ')
E(λ+λ')

Simulation Algorithm

1. Generate a $T_{k+1} - T_k \sim \mathcal{E}(\lambda + \lambda')$
2. Generate a $\xi_k \sim Unif(0, \lambda + \lambda')$
   - 2.1 If $0 < \xi \leq \lambda$ them the green neuron spikes
   - 2.2 If $\lambda < \xi \leq \lambda + \lambda'$ them the red neuron spikes

How about nonhomogeneous Poisson process?





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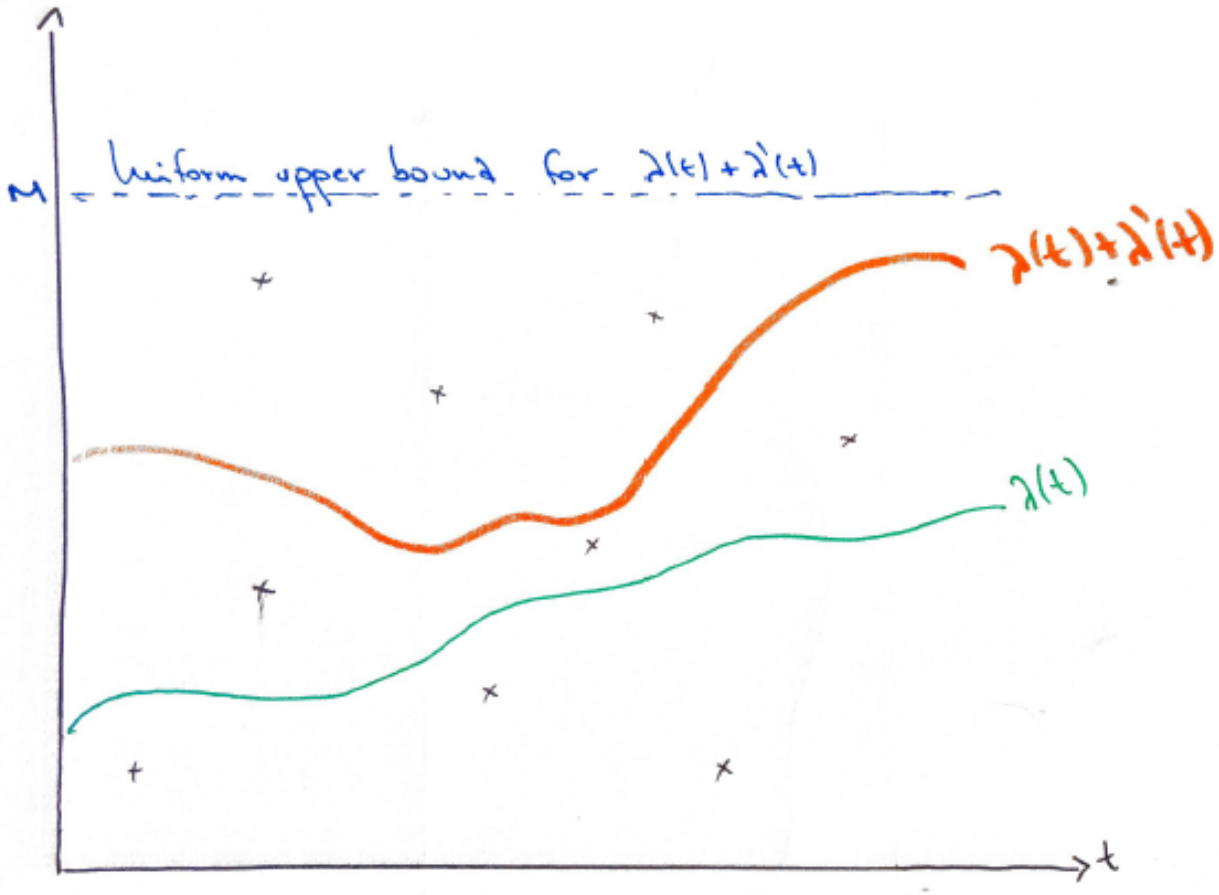

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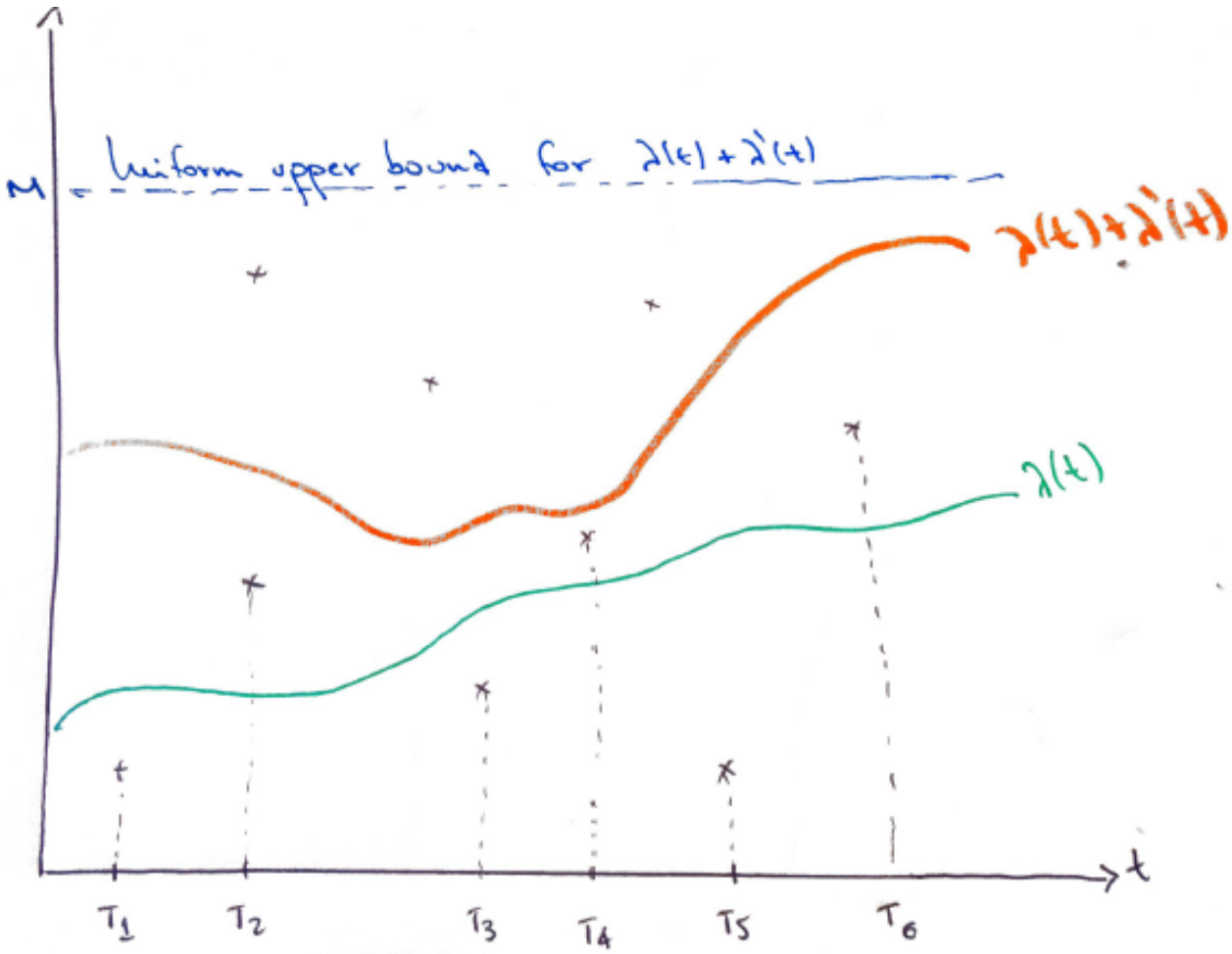
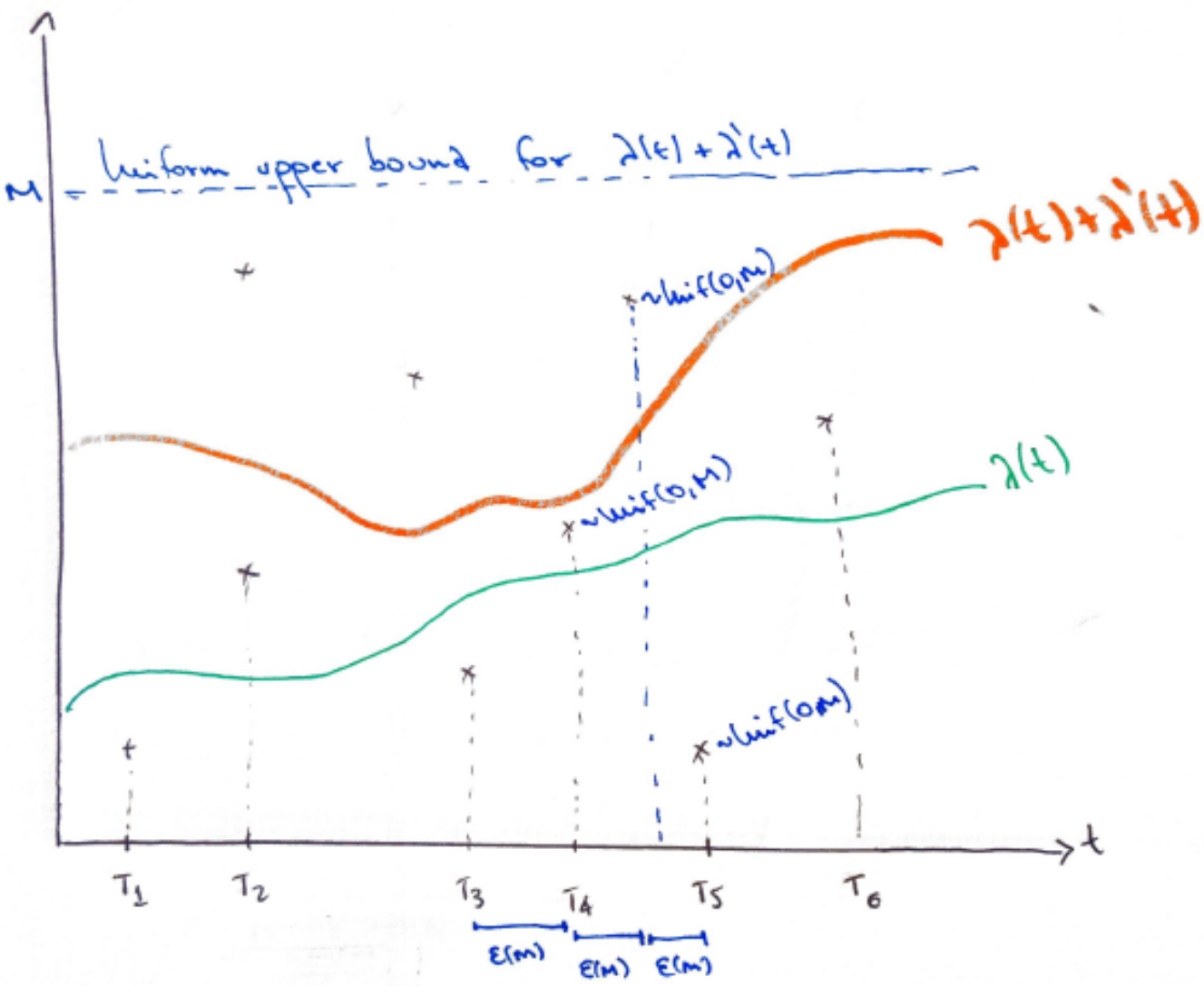


How about nonhomogeneous Poisson process?



Simulation Algorithm

Set $L = 0$

1. Generate a $S \sim \mathcal{E}(M)$
2. Generate a $\xi \sim Unif(0, M)$

   2.1 If $0 < \xi \leq \lambda(S)$ them
   $T_1 = S$
   the green neuron spikes at time T_1

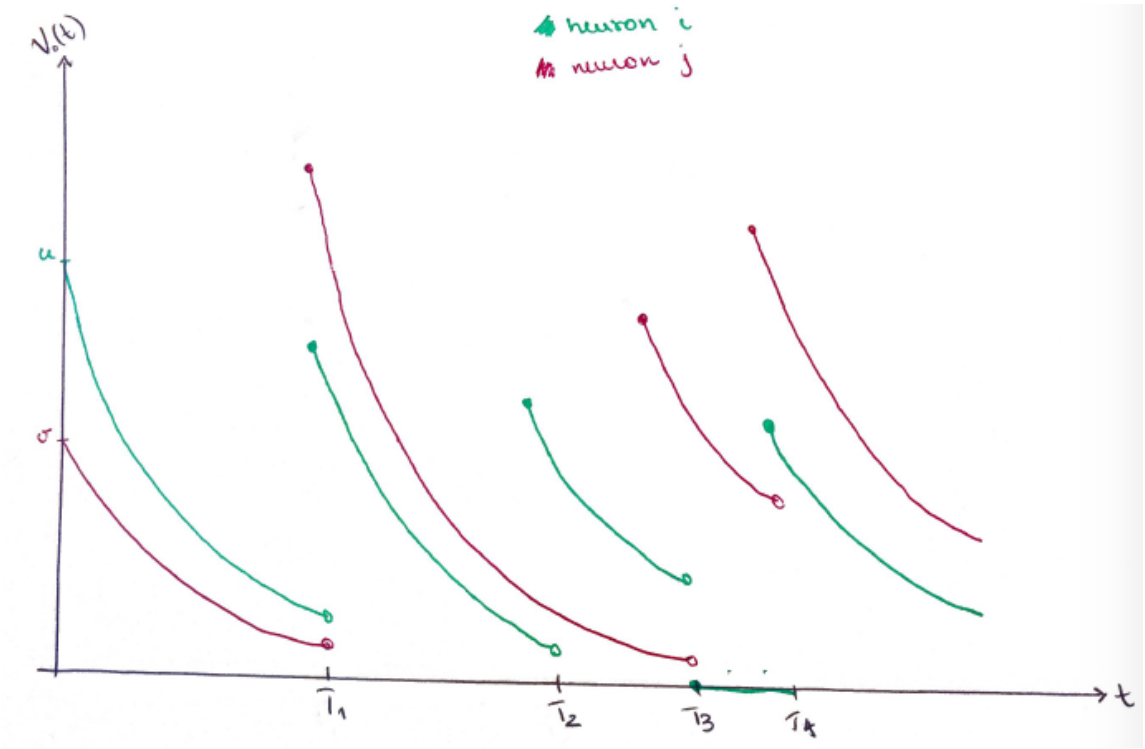
   2.2 If $\lambda(S) < \xi \leq \lambda(S) + \lambda'(S)$ them
   $T_1 = S$
   the red neuron spikes at time T_1

   2.3 If $\lambda(S) + \lambda'(S) < \xi < M$ no one spikes and
   set $L = S$.

How about stochastic IF model??

How about stochastic IF model??

Remember the spiking rate look like



How about stochastic IF model??

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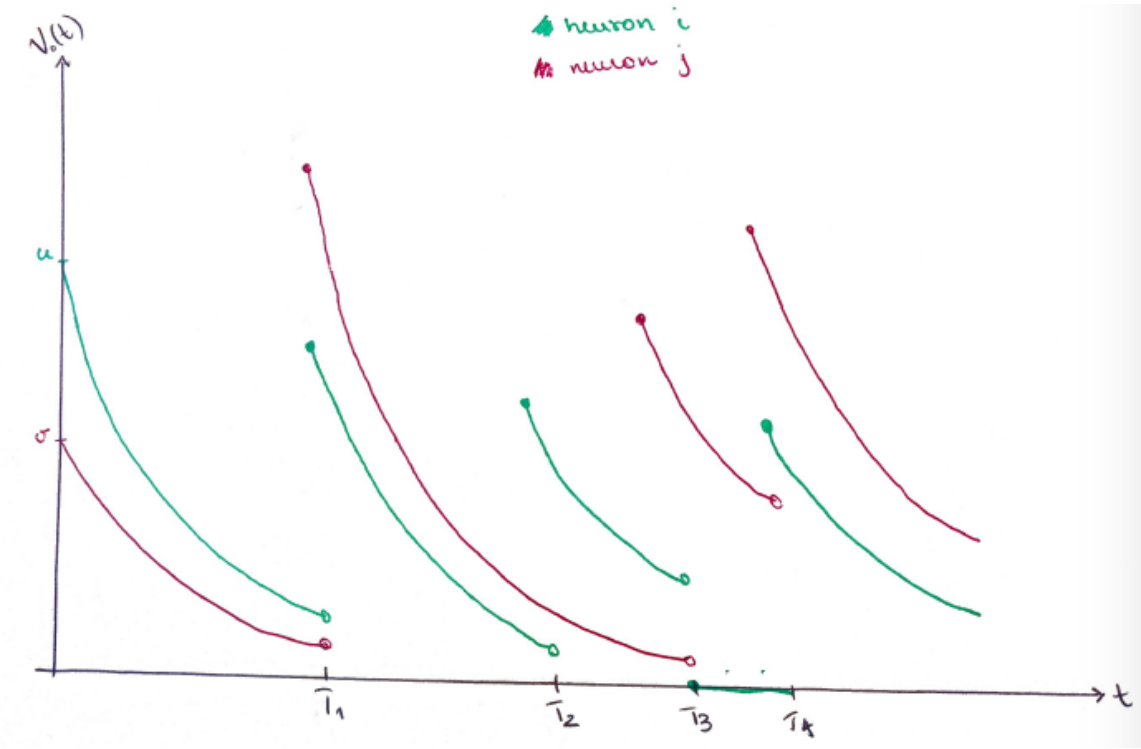
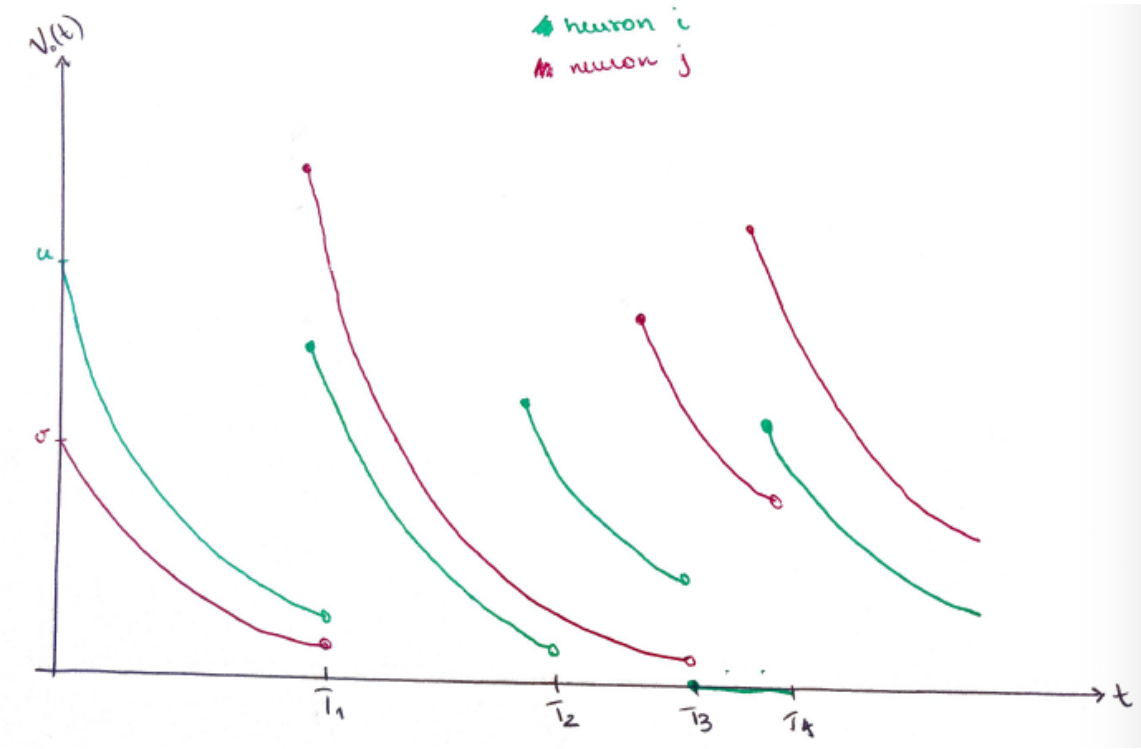


Can we apply the simulation approach for nonhomogeneous Poisson process?

How about stochastic IF model??

Remember the spiking rate look like



Can we apply the simulation approach for nonhomogeneous Poisson process?

Answer: Yes, but piecewisely!!

Simulation Algorithm: consider $\varphi(v) = v$

Given the initial potentials $v_1, \ldots, v_N$. Define $M = \sum v_k$ e set $L = 0$ and $T_0 = 0$.

1. Generate a $S \sim \mathcal{E}(M)$ (number with exponential distribution)
   Update $L = L + S$
2. Generate a $\xi \sim Unif(0, M)$ (number with uniform distribution)
3. Update $v_k = v_k e^{-\alpha S}$
   3.1 If $0 < \xi \leq v_1$ them
   $T_m = T_{m-1} + S$ (accept as a jumping time of the system)
   $V_1(L) = 0$ and
   $V_j(L) = v_j + w_{1 \to j}$, $j = 2, \ldots, N$ (chemical synapses)
   3.2 If $\sum_1^k v_k < \xi \leq \sum_1^{k+1} v_k$ them
   $T_m = T_{k-m} + S$ (accept as a jumping time of the system)
   $V_k(L) = 0$ and $V_j(L) = v_j + w_{k \to j}, j \neq k$ (chemical synapses)
   3.3 If $\sum_1^N v_k < \xi < M$ then no one spikes
   $V_k(L) = v_k \ \forall k$
4. Update $M = \sum_1^N V_k(L)$

Simulation Algorithm: consider $\varphi(v) = v$

Given the initial potentials $v_1, \ldots, v_N$. Define $M = \sum \varphi(v_k)$ e set $L = 0$ and $T_0 = 0$.

1. Generate a $S \sim \mathcal{E}(M)$ (number with exponential distribution)
   Update $L = L + S$
2. Generate a $\xi \sim Unif(0, M)$ (number with uniform distribution)
3. Update $v_k = v_k e^{-\alpha S}$
   3.1 If $0 < \xi \leq \varphi(v_1)$ them
   $T_m = T_{m-1} + S$ (accept as a jumping time of the system)
   $V_1(L) = 0$ and
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Obrigada